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ANALYSIS CONTROL AND DESIGN OF A MECHNICAL SYSTEM SUBJECT TO VIBRATIONS

Miguel Angel Esparza Aranda

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DEDICATION

To the University of San Luis Potosi

To my Teachers

To my Companions

To my Parents and Brothers

To Tita

School of Engineering

Madero y Aldama

Autonomous University of San Luis Potosi

To Candidate Sr. Miguel Angel Esparza Aranda

These Presents.

In response to your request, I am pleased to inform you that the Technical Consultative Council of the School has named as director of the thesis which you should present in your Professional Examination for Electromechanical Engineer, Engineer Francis J. Balch. He has proposed the following theme:

"Analysis, Control, and Design of Mechanical Systems Subject to Vibrations"

I. Introduction
II. Systems Analysis:
    1. Systems with One Degree of Freedom
    2. Systems with Two or More Degrees of Freedom
III. Models
IV. Analogies
V. Impedance and Mobility
VI. Methods for Control of Vibrations
VII. Design of a Mechanical System
VIII. Selection of Equipment
IX. Conclusions

I request that you take note of that fact that in compliance with the specifications of the Law of Professions you must engage in Social Service for a minimum time of at least six months, as an indispensable prerequisite for taking your Professional Examination.

Sincerely,

"Modos et Cunctarum Rerum Mensuras Audebo"

(signed)

Guillermo Labarthe Hernandez, Eng.
Director

December 20, 1966

ABSTRACT

The discussion begins with a definition of vibrations, how they originate and their principle causes as well as their classification for study. The vibrations are analyzed according to this classification and the manner in which they should be studied. Attention is given to different methods used in solving the vibration problems according to the precision required and the time and means available. Known methods that are used to control the vibrations when they can not be avoided in the design due to factors of cost and manufacture, are described. The application of isolators to control the vibrations in an internal combustion machine is discussed. The more important instruments used in the measurement of vibrations are described. Conclusions derived from the study are given.

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CHAPTER I

INTRODUCTION

1.1. General Concepts

Vibration problems occur in almost all areas of engineering, /1* and with the increase of the size and speed of modern machinery these problems become of increasing importance in the area of design in mechanical engineering.

Mechanical vibrations are presently of the highest importance owing to factors such as the following:

(a) Improvement of the technical quality of the product.
(b) Lengthening of the useful life of the product.
(c) Safety during operation.
(d) Increasing efficiency.
(e) Greater precision in manufacture.
(f) Increase with regard to human convenience.

1.2. Source and Cause of Mechanical Vibrations

Every mechanical system which has motion either as a whole or /2 in one of its parts, whether it be rectilinear movement or rotational movement, will be subject to vibrations to a greater or lesser extent depending on the action which causes the motion.

Therefore, vibrations are defined as: the movement of a body or system which is repeated at a given interval of time known as a "period". For this reason, vibrations have periodic motion and the most simple of these motions is the harmonic motion.

The consequences which arise from the vibrations in mechanical systems are as diverse as the following:

(a) Failures due to breaking of parts of the system.
(b) Losses in efficiency and output.
(c) Disagreeable noises and movements.
(d) Lack of precision in measurement.
(e) Lack of comfort.

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* Numbers in margin indicate pagination in original foreign text.

It is true that in many cases vibrations are produced intentionally in many mechanical systems such as the following: electric razors, musical instruments, vibrating sieves, speakers, phonographs, and in the treatment of deaf-mutes, etc.

1.3. Mechanical Systems Subject to Vibration

As has been mentioned, vibrational motions occur to a greater or lesser extent in every mechanical apparatus or structure known to man.

In many cases, the magnitude of these vibrational motions is sufficiently small that they produce no important effects. However, there exist a large number of cases in which vibrations are sufficiently dangerous to cause problems or failures. Several of these failures are encountered in crankshafts of internal combustion engines, in rotating machine shafts, in tools, in turbine engines, in the breaking of springs; they are also of great importance in the destruction of structures such as bridges, buildings, etc., due to the seismic action of the earth as well as of systems subject to vibrations caused by the force of wind. /3

In certain mechanical systems, the failures do not have any immediate effect, as can be seen, but owing to the fatigue of the materials with which the systems are built the failures are caused by the continuous repetition of stresses.

In this connection, it has been noted that sometimes vibrations are produced by natural action (force of wind, seismic motion, etc.); at other times, it frequently happens that vibrations are produced due to the impossibility of fabricating mechanical components which are 100% balanced and homogeneous. In other cases, in which vibrations have an undesirable physiological effect on human beings, or also where they interfere with the operation of delicate instruments, failures do not occur but they produce human discomfort or make impossible a 100% correct functioning of instruments. For these reasons, it is necessary to control vibrational phenomena.

In all of these mechanical systems, whether we are dealing with vibrational motions whose effects are insignificant, such as vibrations which cause inconvenience or the destructive results which they produce, it is necessary to eliminate or reduce their effects to a minimum either through better design or by controlling them. The first step is to make an analysis of vibrational motion covering all their basic characteristics. Only after an investigation of the properties of a vibrating system can a procedure be worked out to obtain the best results.

1.4. Factors Which Enter into Vibrations and Their Classification

A vibration is a movement of a body or systems which is repeated at a given interval of time known as a period. The number of

cycles of motion per unit time is called frequency. The maximum displacement of the body or any part of the system from its position of equilibrium is the amplitude of vibration at this point. /4

There are two general types of vibration: rectilinear and torsional.

Rectilinear vibrations are of two types: vibration along the length which is the axial compression and extension of rods and wires, including the compression and extension of spiral springs; the cross-sectional vibration such as the motion in beams perpendicular to their line of centers.

The amplitude of rectilineal vibrations is usually measured in centimeters.

Torsional vibrations are due to rotational or torsional motion such as that occurring in shafts; their amplitude is measured in radians or degrees.

After this brief classification of the different types of vibrations, they can be subject to different external conditions as if excitation did not exist; for this reason the vibration is called "free." The system will vibrate with decreasing amplitude until it returns to rest. Under other conditions, such as in imbalance of machinery, the body or system can be subject to a periodic excitation in the form of a force or movement; in such a case, the vibration is said to be "forced."

Perhaps this type of vibration is the one encountered most frequently, due to the fact that it is impossible to construct machine components of 100% balance and homogeneity.

If the external excitation frequency in a forced vibration is the same as the natural frequency, the phenomenon known as "resonance" occurs. When this resonance occurs in rotating shafts (due to disequilibrium), the rotational velocity is called "critical velocity."

In Chapter II, it will be seen that the differential equations of motion and their solutions are analogous to the two types of vibrations-rectilineal and torsional. This permits us to simplify the study. Table 1 gives the symbols which represent the different factors entering directly into rectilneal and torsional vibrations. /5

TABLE 1

RECTILINEAR VIBRATIONS		TORSIONAL VIBRATIONS	
TERM	SYMBOL	TERM	SYMBOL
Mass	M, m	Inertia	I. J.
Spring constant	K	Spring Constant	kt
Damping	C	Damping	Ct
Force	F, f	Torsion	T, λ
Displacement	X, x	Angular Displacement	Φ, ϕ
Velocity	$\dot{X}$, $\dot{x}$	Angular Velocity	$\dot{\Phi}$, $\dot{\phi}$
Acceleration	$\ddot{X}$, $\ddot{x}$	Angular Acceleration	$\ddot{\Phi}$, $\ddot{\phi}$

The capital letters represent maximum values or amplitudes; lower-case letters represent instantaneous values.

By a vibrating system is usually meant a combination of elements that, by interaction or the action of external forces, are capable of maintaining a periodic oscillatory motion. In mechanical vibrations, these elements are the mass or inert elements, springs or elastic elements, and resistances or damping elements (dissipators of energy). In a real system, all of these elements are present in every mechanical part of the system.

The three characteristic elements used in a vibrational analysis are defined in the following idealized way:

Mass - A nonelastic solid or an incompressible and nonviscous fluid. It can therefore behave only as an inert element and as such can acquire or lose kinetic energy according to variations in its velocity. /6

Elastic Element - This element does not possess inertia, and it resists deformations or displacements in such a way that the work performed in the production of the deformation or displacement is conserved by the elements in the form of potential energy until the stored work is given up through the return to the initial form or position; that is to say it obeys Hooke's law.

The mass combined with the elastic element constitutes in this idealization a conservation system in which any energy stored in the mass owing to its state of motion (kinetic energy) and any energy contained in the elastic element (spring or equivalent) owing to its formation or displacement (potential energy) can be recovered in its totality. The constant exchange of kinetic energy of the mass for potential energy in the spring is basic to the majority of vibrating systems.

Damping Elements - A damper is a device which dissipates

energy from the vibrating system. The damper causes a force called "damping force" which resists the motion at every instant. The damping force dissipates mechanical energy, which is usually converted into heat, thus draining off the mechanical energy of the system. Dampers which make use of dry friction or Coulomb friction are called friction dampers, and those which use fluid friction as as an energy dissipating force are called viscous dampers.

It should be mentioned that there exist other types of dampers - for example, those which employ electrodynamic damping or damping resulting from air friction. On the other hand, there exists an internal damping in the materials.

The object of design in some cases is the diminution of frictional resistance in machines and instruments, since this resistance dissipates energy and impedes the proper functioning. In other cases, however, additional damping forces are introduced to prevent resonance in the system. The frictional resistance is the type of damping inherent in the materials, and for this reason, many machines do not use other dampers; extremely small energy dissipating forces develop during their operation. For this reason, damping frequently has an insignificant effect on the vibrational analysis, and with the object of simplifying the analysis, these forces are disregarded in many cases. /7

1.5. Degrees of Freedom

A vibrating body is said to vibrate with one degree of freedom when the geometric position of its center of gravity can be defined completely at any instant by a single magnitude. This magnitude can be that of the angle of rotation, of a coordinate, etc.

In general, a vibrating body has as many degrees of freedom as it has coordinates necessary to describe its motion or position; in addition, there are as many types or forms of oscillation in a system as there are degrees of freedom. A body suspended in space has six degrees of freedom, three in the rotational direction and three in the translational direction. A body subject to motion in one plane has four degrees of freedom.

Generally speaking, it can be said that in order to determine the solution for a system in vibration it is necessary to determine first the number of degrees of freedom with which the system vibrates; only thus will we be able to establish its equations of motion.

When one is dealing with a system of one degree of freedom, the solution is solely the equation of motion which governs it. In other words, there will be as many simultaneous equations of motion as there are degrees of freedom in the system.

One ought not to think, however, that a system with one degree of freedom is always very simple. For example, a gasoline engine having 12 cylinders with a rigid crankshaft and with the engine /8

rigidly mounted has only one degree of freedom with all the pistons, connecting rods, valves, cam shaft, etc, in motion. This is due to the fact that a single number (for example, the rotational angle of the crankshaft) completely determines the localization of each moving part of the machine. However, if the machine is mounted on springs so that it can move freely in any direction (as in the case of many modern automobile engines), the system has seven degrees of freedom, six of which belong to the machine as a rigid body in free space, and the seventh is the angle of the crankshaft.

The systems of rigid bodies connected elastically are called concentrated systems or systems of concentrated parameters because they are formed by concentrations of masses joined by spring concentrations without mass. On the other hand, distributed systems are formed by an infinite number of infinitely small concentrations, having distributed mass and elasticity in this form; that is to say, having distributed parameters. Such systems have an infinite number of degrees of freedom and consequently an infinite number of ways in which to vibrate. Beams, plates, fluids, etc. are de facto distributed systems; and, practically speaking, the majority of systems have, to a certain extent, distributed parameters. The motion of a distributed system is governed by one or more partial differential equations, but the equations for which exact solutions are known are relatively few and are limited to more or less idealized systems, as for example, the cross-sectional vibration of a beam.

The thus idealized systems can be studied in one of the following ways:

(1) With the aid of numerical methods for integration.
(2) Experimentation with scale models.
(3) By substituting an arbitrary system of concentrated parameters /9
with the equivalent of a distributed system.
(4) By using electromechanical analogy.
(5) By using mathematical models.
(6) With the aid of the mechanical impedance and the mobility.

As can be seen, there are sufficient means for the solution of vibrating systems; and these can be resolved either directly by one of the above methods or by a combination of these methods.

The direct solution of these differential equations of motion for a vibrating system is tedious, and it is difficult to solve the differential equations simultaneously. This occurs in systems with concentrated parameters or in similar systems; systems of ordinary differential equations then result. For those cases which do not require a high degree of accuracy, solutions can be obtained by approximate methods. For cases requiring a high degree of accuracy, digital computers are employed.

The work of the designer is to select the most appropriate

method, and the solution of almost all vibrational problems is the result of a balance between the desired accuracy and the amount of work.

CHAPTER II

SYSTEMS ANALYSIS

2.1. Systems with One Degree of Freedom

Before entering on a more basic analysis of the problems presented by vibrations it is indespensáble to understand first the development and physical significance of mechanical systems with only one degree of freedom. /10

For this reason a simple system with one degree of freedom will be analyzed. We shall consider first the differential equation of motion which governs it.

A mechanical system is said to have one degree of freedom if its geometric position can be expressed at any instant by a single number.

The systems having one degree of freedom can be subdivided in different ways. One method is to divide them according to the source of energy or force which causes the vibration. Such a classification requires that we consider systems in free vibration and in forced vibration.

We will begin by examining the general case and deriving the equation of motion; from there we will procede to particular cases.

2.1.1. Equation of Motion for a System with One Degree of Freedom.

We consider a mass rigidly suspended from the ceiling by means of a spring as shown in Figure 1. The "rigidity" of the spring is calculated through its spring constant designated by K, which, for purposes of definition, can be considered the quantity of tensor force necessary to extend the spring one displacement unit. Between the body and the rigid wall there is also a damping mechanism of air or oil, which for the case under consideration, can be either the specific resistance of the body or simply the friction. This damping, or better the damping force, is directly proportional to the velocity and in a direction opposite to it. Mathematically stated: /11

Damping force

$$\alpha \; v \qquad v \triangleq \frac{dx}{dt} = \dot{x}$$

$$F_A = C \frac{dx}{dt} = C \dot{x}$$

where C represents the damping constant or viscous damping coefficient.

For the purpose of standardizing and making representative the elements which enter into a vibrating system, we designate by the following symbol:

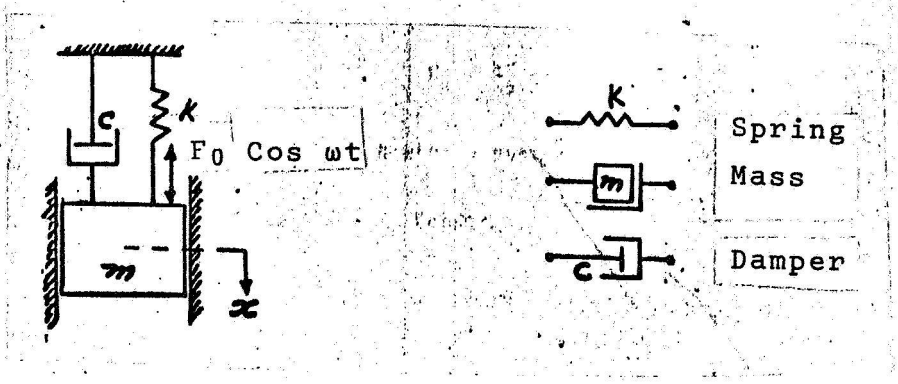


Fig. 1.

An external alternating force of value $f = F_0 \cos \omega t$ acts upon /12 this system (this type of force is considered to be the most frequently encountered) which is produced by a mechanism which we shall not specify.

The problem consists of calculating the motion of the mass m due to this external force. Or, in other words, if x is the distance between any position of the mass during its movement and the equilibrium position, we are required to find x(t) or displacement as a function of time.

In order to calculate this equation of motion, let us assume that:

All the forces which act on the mass will be regarded as positive when directed downwards and negative when directed upwards.

The sign of spring force is negative because the spring pulls the mass m upwards when the displacement is downward; or the spring force is negative when x is positive.

Then $F_R = -K\dot{x}$.

The damping force on the mass is also negative since it is directed opposite to the velocity $\dot{x}$; it acts upward (negative) while $\ddot{x}$ acts downward (positive).

$$F_A = - C\dot{x}$$

The forces which act on the mass are therefore:

$$F_A + F_R + F_0 \cos\omega t$$

Following Newton's law and substituting:

$$m\ddot{x} = - Kx - C\dot{x} + F_0 \cos\omega t \qquad \ddot{x} \triangleq \frac{d^2x}{dt^2}$$

$$\therefore \; m\ddot{x} + C\dot{x} + kx = F_0 \cos\omega t \qquad (1)$$

This equation is very important and is known as the differential equation of motion of a system with one degree of freedom. /13

The four terms of equation (1) are:

$m\ddot{x}$ = inertial force
$C\dot{x}$ = damping force
kx = spring force
$F_0 \cos\omega t$ = external force

Before proceeding to find the solution of equation (1), we will consider the system of torsional vibration with one degree of freedom; this will help us understand the similarity of equations and will serve in the future to establish the analogies:

Figure 2 shows a disc of moment of inertia I fixed to a shaft of torsional rigidity K_t, which is defined as the torque necessary to produce a radian of torsion on the disc.

We will consider the torsional motion of the disc under the external influence of a couple applied to it of value $T_0 \cos \omega t$. This, again, yields a system with one degree of freedom since the torsional displacement of the disc from its equilibrium position can be expressed by a single quantity, the angle ϕ.

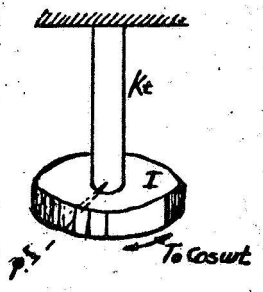


Fig. 2

The application of Newton's law to this rotational system leads to: /14

Torsion = Moment of inertia × angular acceleration

$$T = I \frac{d^2\phi}{dt^2} = I\ddot{\phi}$$

In this system there are three torques or torsions acting on the disc: spring torsion, damping torsion, and external torsion.

Spring torsion = $-k_t \phi$

Damping torsion = $- C_t \frac{d\phi}{dt} = - C_t \dot{\phi}$

C_t = Damping constant

External torsion = $T_0 \cos\omega t$

Applying Newton's law, this yields:

$$I\ddot{\phi} = - kt\,\phi - C_t \dot{\phi} + T_0 \cos\omega t$$

$$I\ddot{\phi} + C_t \dot{\phi} + k_t \phi = T_0 \cos\omega t \qquad (2)$$

It can be seen that this equation is of the same form as Equation (1).

From this similarity we are led to believe that the solution for a system with rectilinear vibrations is the same as that for a system subject to torsional vibrations, with the exception that the constant values must be substituted because they are different for both systems.

The equations which have been established up to this point correspond to the general equation of motion for a system with one degree of freedom acted upon by an external force.

As is known, the equation, whether it be (1) or (2), is a linear differential equation of the second degree and its solution contains a particular integral and a complementary integral.

The complementary function is observed when the right-hand side /15 of equations (1) or (2) is zero. In our system, this corresponds to making $F_0 \cos\omega t = 0$ and is equivalent to saying that the system is not excited by external force; this corresponds to a system with free damping and is therefore a transitory vibration in a real system.

The particular solution or function corresponds to an induced vibration and is equivalent to a constant vibration.

For this reason the study is subdivided into systems of vibrations with free damping and systems of vibrations with forced damping. However, the undamped system of vibrations can also be included; this, however, is a case which will not be studied since, neglecting the equivalent damping for an ideal system, any vibrating system --however small--has damping.

The damping action can be caused by a great variety of effects,

such as the following:

(a) Internal friction of the material.
(b) By the action of a fluid.
(c) By magnetic fields.
(d) By dry friction.

Taking the preceding into consideration we can begin to study the free vibrations and following these, forced vibrations.

2.1.2. Damped free vibrations

Free vibrations exist when no external force acts upon the system, or when we have in equation (1):

$$m\ddot{x} + C\dot{x} + kx = 0 \tag{3}$$

The solution of this equation is:

$$x = B_1 e^{P_1 t} + B_2 e^{P_2 t} \tag{4}$$

where /16

$$P_1 = -\frac{C}{2m} + \sqrt{\left(\frac{C}{2m}\right)^2 - \frac{k}{m}} \tag{5}$$

$$P_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{6}$$

letting B_1 and B_2 be arbitrary constants, which we can determine if we know the initial conditions.

There are three values of P_1 and P_2 which we must consider:

(1) $\left(\frac{c}{2m}\right)^2 > \frac{k}{m}$ is known as overdamped vibration.

(2) $\left(\frac{c}{2m}\right)^2 < \frac{k}{m}$ is known as periodic damping.

(3) $P_1 = P_2$ or $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} =$ is known as critical damping.

2.1.2.(a). Critical Damping. The state of critical damping is important because it provides us with a method of measuring the relative quantity of damping of a particular system. In practice critical damping rarely occurs since it is difficult to maintain the exact relations required in the passive elements in order to produce it.

For purposes of differentiation the critical damping factor is designated as Cc, and the expression for the real quantity of damping C for the critical damping Cc is the relative damping quantity of the system; this expression is designated by ζ:

$$\zeta = \frac{c}{Cc}$$

With critical damping we have: /17

$$\left(\frac{Cc}{2m}\right)^2 = \frac{k}{m} \quad \text{(condition for } P_1 = P_2 = P = \frac{Cc}{-2m}\text{)}$$

Equation (4) can be written as:

$$x = B\, e^{pt} = B\, e^{-(Cc/2m)t} \quad \text{+ letting}$$

$$B = B_1 + B_2$$

but this is not the solution of equation (3) since, being of the second degree, it should contain two integration constants.

The solution of (3) is:

$$X = B_3\, e^{-(Cc/2m)t} + B_A t\, e^{-(Cc/2m)t} \qquad (7)$$

which satisfies the condition when both roots of (3) are equal ($P_1 = P_2$).

The general initial conditions are:

$x = x_0$, $v = v_0$ for $t = 0$ whence one obtains the values of B_3 and B_4:

$$B_3 = x_0 \qquad B_A = v_0 + \left(\frac{Cc}{2m}\right) x_0$$

For critical damping, Equation (7) will then be:

$$x = x_0\, e^{-(Cc/2m)t} + \left(v_0 + \frac{Cc}{2m}\, x_0\right) t\, e^{-(Cc/2m)t} \qquad (8)$$

Since for the critical damping conditions we have:

$$\left(\frac{Cc}{2m}\right)^2 = \frac{k}{m} \text{ we obtain } Cc = 2\sqrt{mk}$$

and since the natural frequency $\omega_n = \sqrt{\frac{k}{m}}$ we obtain the following important relation:

$$Cc = 2m\omega_n = \frac{2k}{\omega_n} \qquad (9)$$ /18

2.1.2.(b). Overdamping Overdamping is possible when $\left(\frac{C}{2m}\right)^2 > \frac{k}{m}$, y - , and this is associated with a relatively large damping factor C. These types of vibrations are termed non-periodic because the mass returns slowly to its equilibrium position.

Since $(c/2m)^2 > \frac{k}{m}$, the values of P_1 and P_2 in (5) and (6) are always real but negative values.

Let us again consider the initial conditions $x = x_0$, $v = v_0$ for $t = 0$. For (4) we find the values for B_1 and B_2:

$$B_1 = \frac{x_0 P_2 - v_0}{P_2 - P_1} \qquad (10)$$

$$B_2 = \frac{v_0 - x_0 P_1}{P_2 - P_1} \qquad (11)$$

and substituting (10) and (11) in (4) we obtain:

$$x = \frac{1}{P_1 - P_2}\left[(v_0 - x_0 P_2)\, e^{P_1 t} + (x_0 P_1 - v_0)\, e^{P_2 t}\right] \qquad (12)$$

Equation (12) expresses the displacement x as a function of time with overdamping and general initial conditions. This equation can be transformed into a dimensionless equation by substituting the value $2m = Cc/\omega_n$ found in (9) and that of (5) and (6) in (12); by means of simple mathematical transformations and assigning the value $\zeta = C/Cc$ one obtains:

$$x = \frac{e^{-\zeta\omega_n t}}{2\omega_n\sqrt{\zeta^2-1}}\Big(\left[v_0 - x_0\omega_n\left(-\zeta - \sqrt{\zeta^2 - }\right)\right]. e^{\omega_n\sqrt{\zeta^2-1}\, t} + \left[x_0\omega_n \left(-\zeta + \sqrt{\zeta^2-1}\right) - v_0 \right] e^{-\omega_n\sqrt{\zeta^2-1}\, t} \Big) \qquad (13)$$

If the natural frequency ω_n without damping is known, the response curve can be graphed for the different values of ζ. /19

Figure 3 shows a family of curves for a system in which $\omega_n = 1$; it must be considered in addition that the mass does not have initial velocity and that the initial displacement x_0 is unity.

From this figure it can be observed that the curve falls with critical damping ($\zeta = 1$). If the damping factor is infinite (Cor $\zeta = \infty$) the mass comes to a halt and the curve becomes horizontal. When the amount of damping is reduced, the mass returns to its equilibrium position more quickly, and--in the limited by critical damping--the return is much more rapid.

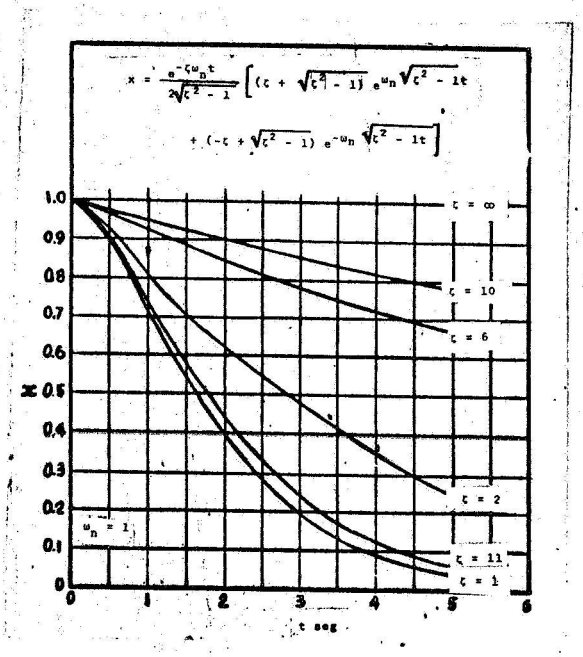


Fig. 3.

2.1.2.(c). Periodic Damping /20

Periodic damping in free vibrations occurs when $(C/2m)^2 < k/m$ or the expression $\zeta < 1$.

The mass vibrates with a diminution of its amplitude as is shown in Figure 4.

Since $(C/2m)^2 < k/m$, the exponents P_1 and P_2, in agreement with (5) and (6), are complete quantities; resolving equation (4) one obtains for this case

$$x = e^{-(C/2m)t}\left[(B_1+B_2)\cos\omega_{nd}t + (jB_1-jB_2)\sin\omega_{nd}t\right] \qquad (14)$$

letting $\omega_{nd} = \sqrt{k/m - (C/2m)^2}$, which is the natural damping frequency.

By considering the initial conditions $x = x_0$, $\dot{x} = v_0$ for $t = 0$, one finds the values for B_1 and B_2.

$$B_1 + B_2 = x_0 \; ; \; jB_1 - jB_2 = (1/\omega_{nd})(v_0 + Cx_0/2m)$$

Substituting this in equation (14) one obtains:

$$x = e^{-(c/2m)t}\left[x_0\cos\omega_{nd}t + \frac{1}{\omega_{nd}}\left(v_0 + \frac{C}{2m}x_0\right)\sin\omega_{nd}t\right] \qquad (15)$$

An examination of Equation (15) shows that the terms within brackets can be represented by vectors revolving at the same angular frequency ω_{nd}. Both vectors are 90°, since one is a cosine function and the other is a sine function. The first vector has the magnitude x_0 and the latter the magnitude $(1/\omega_{nd})(\nu_0 + Cx_0/2m)$. This is shown in Figure 5; the resultant vector will be: /21

$$\sqrt{x_0^2 + [(\nu_0 + Cx_0/2m)/\omega_{nd}]^2}$$

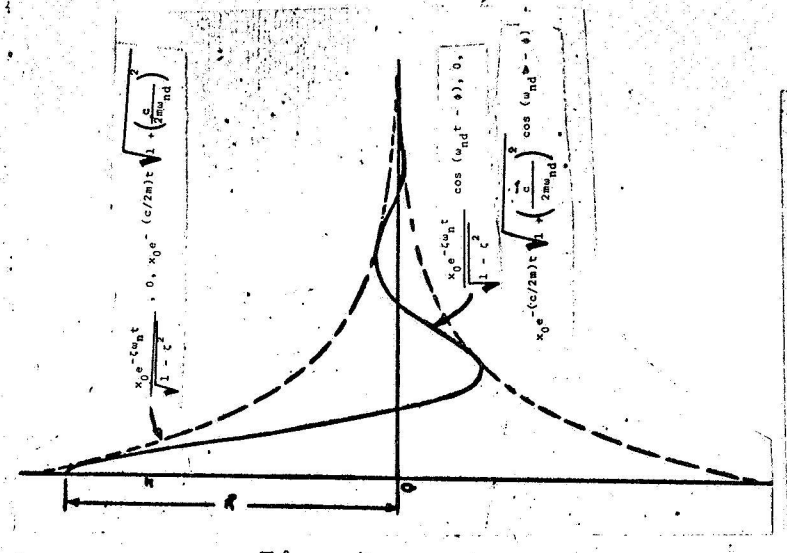


Fig. 4.

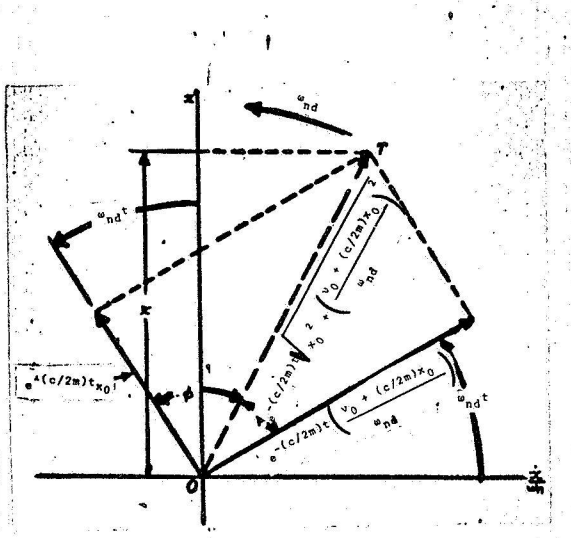


Fig. 5.

The lag of the vector cosine is given by the phase constant angle ϕ or the progress of the vector sine is given by the phase angle ψ where: /22

$$\tan\phi = (\nu_0 + Cx_0/2m)\,\omega_{nd}\,x_0 \quad , \quad \tan\psi = 1/\tan\phi$$

Equation (15) can be written in either of the two following forms:

$$x = e^{-(C/2m)t}\sqrt{x_0^2 + [(\nu_0 + Cx_0/2m)\,\omega_{nd}]^2}\cos(\omega_{nd}t - \phi) \qquad (16)$$

$$x = e^{-(C/2m)t}\sqrt{x_0^2 + [(\nu_0 + Cx_0/2m)/\omega_{nd}]^2}\sin(\omega_{nd}t + \psi) \qquad (17)$$

Figure 4 indicates the declination of the curve for the initial conditions $x = x_0$, $\dot{x} = \nu_0 = 0$ when $t = 0$. The modification of (16) for this special case is given in the figure. It can be seen that the exponential term reduces the amplitude of the vibrations when

the time is increased. Although the magnitudes of the rotational vectors (Fig. 5) remain constant, the resultant vector does not; it continues to decrease in agreement with the exponential since the extreme of this vector describes a spiral exponential. Equations (15) and (16) contain two factors, the first of which is an exponential term similar to the conditions of overdamping and the second is a harmonic vibratory motion. The resulting curve of motion is a sinusoidal curve whose amplitude decreases with time according to the term $e^{-(C/2m)t}$. The broken line represents the envelope of the amplitude, which continues to expand for small values of damping C. It can also be seen from (15) and (16) that the damping frequency is less than the frequency without damping. The expression lying between these frequencies (in terms of the expression of damping ζ) can be found by means of (9): $C_c^2 = 4mk$

/23

$$\omega_{nd}/\omega_n = \left(\sqrt{k/m - (C/2m)^2}\right)\Big/\sqrt{k/m} = \sqrt{1-\zeta^2} \qquad (18)$$

which is the equation of a circle.

Equation (16) can be transformed into a dimensionless equation by the use of Equations (9) and (18).

$$x = e^{-\zeta\omega_n t}\sqrt{x_0^2 + (v_0 + \zeta\omega_n x_0)^2/\omega_n^2 (1-\zeta^2)}\,\cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi) \qquad (19)$$

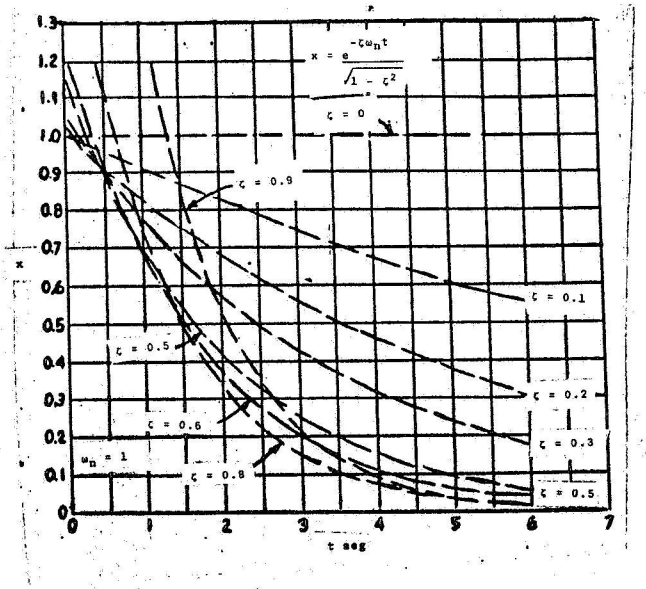


Fig. 6

The envelopes for the superior side of the sinusoidal curve for these initial conditions are based on the first factor of Equation (19) and are graphed for $\omega_n = 1$ and various values of ζ in Figure 6. If there is no damping $\zeta = 0$, the vibration does not stop, the envelope curve will be a horizontal line.

/24

On some occasions one wishes to find the way in which the amplitudes of the periodic vibration decreases and which supplies a measure for the expression for reduction.

Let tn be the time elapsing from the start of the free vibrations at any point x_n of the envelope curve, and let t_{n+1} be the time elapsing from the beginning of the free vibrations at another point x_{n+1} of this curve after one period. The time elapsing between these two points is equal to the period $2\pi/\omega_{nd}$; or let

$$t_{n+1} = t_n + \frac{2\Pi}{\omega_{nd}}$$

From the exponential factor of Equation (16):

$$x_n = e^{-(C/2m)\, t_n}$$

$$x_{n+1} = e^{-(C/2m)t_{n+1}} = e^{-(C/2m)\, tn\, +\, 2\Pi/\omega_{nd})}$$

Taking the expression x_{n+1}/x_n: /25

$$\frac{x_{n+1}}{x_n} = \frac{e^{-(C/2m)\,(t_n+2\Pi/\omega_{nd})}}{e^{-(C/2m)t_n}} = e^{-(\Pi C/m\omega_{nd})}$$

It is obvious from this equation that the amplitude decreases in constant geometric proportion which is a function of C, m and ω_{nd}. The term $\Pi C/m\omega_{nd}$ is known as "logarithmic decrement" and is designated by the letter δ:

$$\delta = - \ln \left(\frac{x_{n+1}}{x_n}\right) = \ln \left(\frac{x_n}{x_{n+1}}\right)$$

The logarithmic decrement is therefore a convenient measure of decline of a free vibration and a measure of the quantity of damping in a system. This can be expressed as a function of ζ with the aid of Equation (9).

$$\delta = \Pi C / m\omega_{nd} = \Pi C/m \sqrt{k/m - C^2/4m^2} = 2\Pi\zeta \Big/ \sqrt{1-\zeta^2} \qquad (20)$$

or

$$\zeta = \frac{\delta}{\sqrt{4\Pi^2 + \delta^2}} \qquad (21)$$

It is interesting to note in this last equation that $\zeta = \frac{1}{2\Pi}$ when the logarithmic decrement is unity.

2.1.3. Forced Vibrations.

By a system with forced vibrations is meant a system excited by a force or motion. This excitation can be developed within the system itself as in the case of an imbalance of a rotating member of the system or developed by external means, e.g., the vibration of nearby machinery which is transmitted to the system being considered.

The excitation can be periodic in the form of a simple harmonic motion or can have any other periodic in the form of a simple harmonic motion or can have any other periodic form which can be resolved into harmonic components which are expressed in the form of a Fourier series. A third type of repeated excitation is in the

form of a series of shocks or pulses, waves of pulses, or pulse functions of force or displacement applied to the mass or support.

2.1.3.(a) Sources of Excitation. The most common source of internal excitation is probably imbalance. In Chapter VI we will give a practical method for balancing.

The principle sources of excitation are:

(a) Thermal effects: Frictional heating between two members of a system having different expansion coefficients gives rise to imbalance; or during operation, the crystalline structure of a metal can be damaged.

(b) Resonance of a unit or a part.

(c) Defects or loosening of the fittings.

(d) Disalignment or other faults in the assembly or installation.

(e) Warping of rods.

(f) Magnetic effects.

2.1.3.(b). Forced Damped Vibrations. Equation (1) shows the displacement as a function of time of the system given in Figure 1 whose solution is found immediately, as has been noted; Equation (1) contains the complementary integral given in Equation (4) /27

$$x_c = B_1 e^{P_1 t} + B_2 e^{P_2 t}$$

Or for conditions of periodic damping, Equation (14) can be written thus:

$$x_c = e^{-(C/2m)t} (B_3 \cos\omega_{nd} t + B_4 \sin\omega_{nd} t) \qquad (22)$$

Each of these solutions is transitory based on free vibrations.

The particular integral is based on the force term whose solution is:

$$x_p = \frac{F\ [(k-m\omega^2)\cos\omega t + (c\omega)\sin\omega t]}{(k-m\omega^2)^2 + (C\omega)^2} \qquad (23)$$

This solution can be simplified by using Equation (9) in the form $k = Cc\omega_n/2$ and by the introduction of two new terms $\beta = \omega/\omega_n$ which is the expression of the frequency imprinted on the natural frequency and $X_{st} = F/k$. The term X_{st} is the deflection of the system owing to the maximum imprinted force acting as a static charge. Dividing the numerator and denominator of Equation (23) by K_2 and making the necessary substitutions:

$$x_p = \frac{X_{st}\ [(1-\beta^2)\cos\omega t + 2\zeta\beta\sin\omega t]}{(1-\beta^2)^2 + (2\zeta\beta)^2} \qquad (24)$$

This equation consists of two terms, one of which contains cos ωt, and the other sin ωt. Both can be considered perpendicular /28 vectors and can be considered to be rotating at the angular velocity ω as is shown in Figure 7. From this figure it can be deduced that the vector sum is:

$$X_p = \frac{X_{st}}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}$$

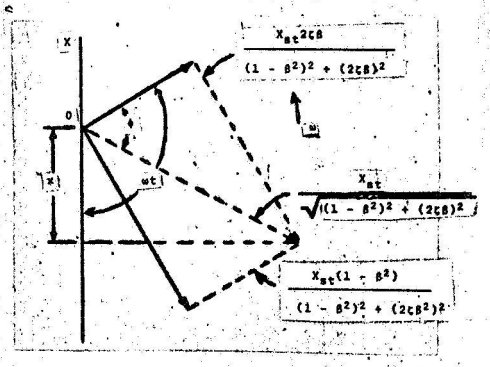


Fig. 7

In the light of this last equation, Equation (24) can be transformed to an equation of the form:

$$x_p = \frac{X_{st}}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} \cos(\omega t - \phi) \qquad (25)$$

where $\tan\phi = 2\zeta\beta/(1-\beta^2)$. (26)

The complete solution of Equation (1) can be obtained by summing the complementary integral Equation (22) and the particular integral Equation (25):

$$x = e^{-(C/2m)t}(B_3 \cos\omega_{nd}t + B_4 \sin\omega_{nd}t) + \frac{X_{st}}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} \cos(\omega t - \phi) \qquad (27)$$ /29

The exact form of the curve resulting from Equation (27) will depend on the initial motion conditions.

The mathematical solution can be somewhat difficult to visualize and a clear understanding of the action after transition, since this disappears, can be obtained from a vectorial solution. The motion of the mass is simple harmonic with a frequency equal to the impressed frequency ω although all the forces which act on the mass can be represented by rotational vectors.

It is noted that we can differentiate a rotational vector x from an angular velocity ω by multiplying its magnitude by ω and rotating forward (in the direction of rotation) 90°.

In Figure 8, the motion of the mass is given in the vertical direction (see Fig. 1). The instantaneous displacement of the mass is represented by the projection on the vertical axis x o x . The vector force is represented by solid lines and the instantaneous values of these forces which act on the mass are represented by the projections on the axis x o x . Following from this discussion,

the forces are:

(a) Spring force of maximum value KX, acting in the direction opposite to the displacement X. Its instantaneous value is K_x, and its action is upward. /30

(b) The damping force is CωX and is based on the first derivative of the displacement Cx and is therefore 90° ahead of the spring force. Its projection on axis x o x is Cωx and its action is downward.

(c) The inertial force $m\ddot{x}$ is the second derivative of the displacement, and acts 180° ahead of the spring force; its projection on the axis x o x is $m\omega^2 x$ and its action is downward.

(d) The force vector of the diagram is the impressed force (of excitation) F and occupies a position such that the system will be in dynamic equilibrium. In order to satisfy this condition, it acts ϕ degrees ahead of the displacement X so that the sum of the forces perpendicular and parallel to the displacement X are balanced. The relation for the forces parallel to X are: /31

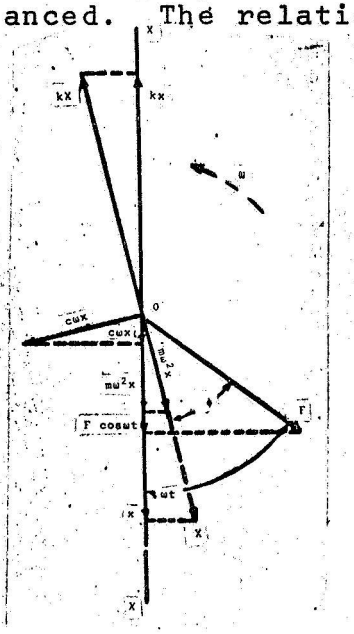


Fig. 8.

$$KX - m\omega^2 X - F\cos\phi = 0$$

$$\cos\phi = \frac{X(k - m\omega^2)}{F} \qquad (28)$$

For the forces perpendicular to the axis x:

$$\sin\phi = \frac{C\omega X}{F} \qquad (29)$$

Since

$$\tan\phi = \sin\phi / \cos\phi = C\omega / (k - m\omega^2) \qquad (30)$$

and since $K = m\omega_n^2$ and by Equation (9) $Cc = 2m\omega_n$, we then obtain:

$$\tan\phi = 2\zeta\beta / (1-\beta^2) \qquad (31)$$

which is the same as Equation (26).

It is apparent that the phase angle ϕ depends on the quanity of damping ζ and the frequency ω of the impressed force or β. This relation is given graphically in Figure 9. It can be observed that if there is no damping ($\zeta = 0$) the angle is instantly ruptured from 0 to 180° for the natural frequency ($\beta = 1$). When the quan-

tity of damping is increased, the change takes place more gradually as is shown in the figure.

An expression for the amplitude of the mass at an unpressed frequency can be found by substituting Equations (28) and (29) in the trigometric relation $\sin^2\phi + \cos^2\phi = 1$; this gives us: /32

$$\frac{C^2\omega^2X^2}{F^2} + \frac{X^2(K-m\omega^2)^2}{F^2} = 1$$

Solving for X:

$$X/X_{st} = \frac{F}{\sqrt{(C\omega)^2 + (k-m\omega^2)^2}} \qquad (32)$$

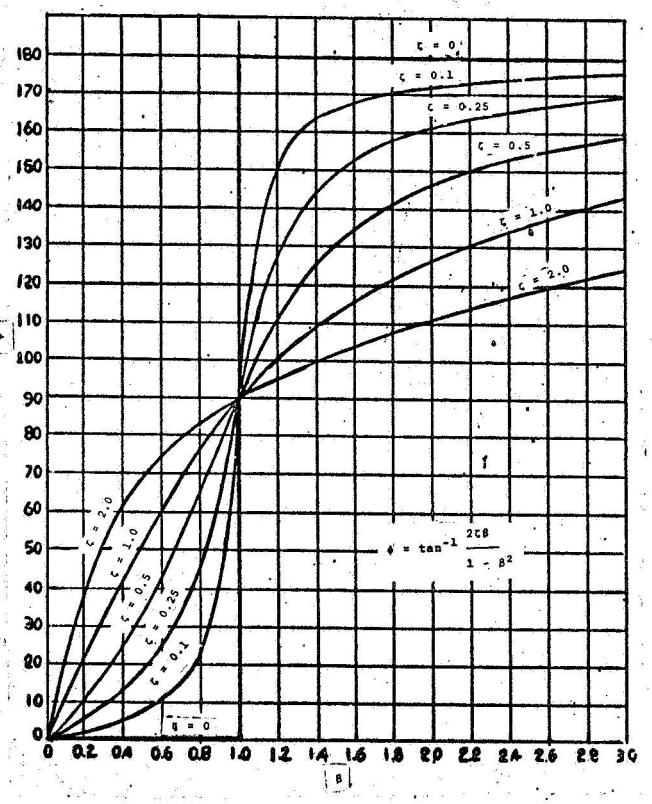


Fig. 9

This equation can be transformed into a dimensionless equation by introducing the terms previously defined: /33 $X_{st} = F/K$ and $Cc = 2m\omega_n$; one obtains:

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{(2\zeta\beta)^2 + (1-\beta^2)^2}} \qquad (33)$$

Equation (33) gives the values for the amplitude of maximum displacement X in terms of β, ζ and X_{st} but does not give an expression for instantaneous values of displacement x as a function of t. Examination of Figure 8 shows that the displacement is:

$$x = X\cos(\omega t - \phi)$$

$$x = \frac{F}{\sqrt{(C\omega)^2 + (K-m\omega^2)^2}}\cos(\omega t - \phi) \qquad (34)$$

Since it is possible to consider the quantity of damping in various systems as insignificant, it is sometimes desirable to consider the case in which C or ζ is zero and to write the equations conforming with Equations (34) and (35). For impressed frequencies

lower than the natural frequency of the system, the impressed force is in phase with the displacement; and, since $\phi = 0$,

$$x = [\, F / (\, K - m\omega^2 \,) \,] \cos\omega t = [\, X_{st} / (\, 1 - \beta^2 \,) \,] \cos\omega t \qquad (36)$$

For impressed frequencies greater than the natural frequency of the system, the impressed force is 180° out of phase with the displacement (Fig. 9); Equation (36) will be negative since $\cos(\omega t - 180^\circ) = -\cos\omega t$.

In Figure 10, the expression of amplitude X/X_{st} is plotted against the frequency β for various quantities of damping ζ. /34

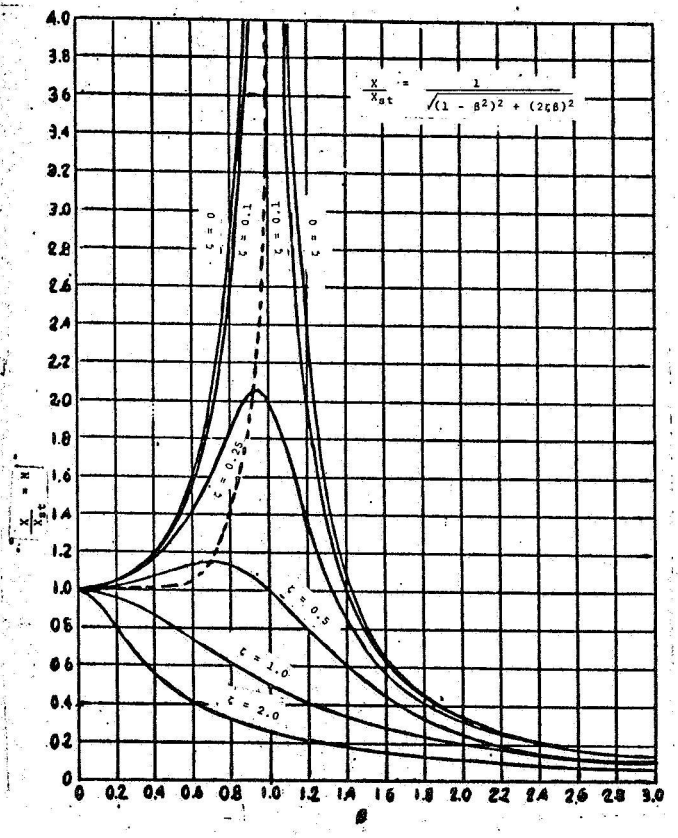


Fig. 10

When a particular value of F acts on a system having fixed values for m, K, and C, the value of ω_n and X_{st} are fixed. Then the graph in Figure 10 can be considered a graph of the amplitude of the mass against the impressed frequency. It is observed that greater quantities of damping cause reductions in amplitude--which is to be expected.

The arrangement of curves in Figure 10 is called "amplification factor" M since it is the expression of real amplitude of the mass at a given frequency for the deflection which obtains if the impressed force is applied in a static charge.

When the impressed frequency is low, the velocity and acceleration of mass are small, since the damping forces and the inertial forces are small. In this case, the mass will vibrate at an amplitude of approximately X_{st} or F/K, that is X/X_{st} is almost unity as is shown in Figure 10. The phase angle ϕ is close to zero as is shown in Figure 9. /35

It can be seen from Figure 10 that the peak value of X/X_{st} or M does not occur when $\beta = 1$ and when damping exists. It is of interest to know the value of β when the peak occurs as a function

of ζ. It can be found by expressing Equation (33) in the form:

$$M = f\left(1 - 2\beta^2 + \beta^4 + 4\zeta^2\beta^2\right)^{-1/2}$$

and finding its maximum value, one receives:

$$\frac{dM}{d\beta} = \frac{-4\beta + 4\beta^3 + 8\zeta^2\beta}{-2\left(1-2\beta^2+\beta^4+4\zeta^2\beta^2\right)^{3/2}} = 0$$

whence,

$$\beta = \sqrt{1 - 2\zeta^2} \qquad (37)$$

This curve is given as a broken line in Figure 10.

2.2. Systems with Two and More Degrees of Freedom

2.2.1. Introduction

It has been mentioned in the preceding chapter that many complex vibrating systems are made up of a system or group of systems less complex than the original for the purpose of facilitating and, in some cases, of making possible the analysis of vibrational phenomena. Thus, also has the theory of systems with one degree of freedom been discussed, in spite of the fact that it is an exactly idealized system which is rarely encountered in reality. Although in some cases it is too idealized, it is without question, of great usefulness for the explanation and analysis of the phenomena of measurements, for the calculation of natural frequency and the explanation by this theory of the functionings of many instruments for the measurement of vibrations. /36

However, this exhausts almost all the possibilities of application, and with the object of being able to explain several additional phenomena it is necessary to develop the theory of more complicated systems which in turn cannot be reduced to systems of one degree of freedom if one wishes to determine the properties which refer frequently to internal relations of the system without which the analysis falls short of reality.

When there is more than one degree of freedom, the complexity of the problem increases with the increase in the number of degrees. The frequency of the exciting force must be considered in relation to each one of the natural frequencies and, as a result of this, the "aspects of response" are more complicated as compared to systems of one degree of freedom.

For a system of two degrees of freedom, the number of equations necessary to resolve the system is two, since the possibility for motion of two masses is now of two distinct forms; that is to say, in phase or in opposition. The problem which must be solved for the solution of a system of n degrees of freedom does not differ much for the two equations; it is for this reason that several methods are developed for continuation for solutions of systems with n degrees of freedom.

With the object of forwarding the development of mechanical systems such as electrical machinery, internal combustion engines, etc., and making use of the fact mentioned in the previous chapter that there is no difference between the solutions for systems with linear vibrations and systems with torsional vibrations, and since in the above-mentioned systems torsional vibration occurs more frequently, more importance will be attached to systems with torsional vibrations. Generally, this occurs in moving parts of the machine such as shafts, crankshafts, etc. The study is divided into free and forced torsional vibrations in shafts, and later in crankshafts. /37

2.2.2. Free Torsional Vibrations of Shafts.

The general case for vibrations in shafts with various rotating masses will be analyzed (see Fig. 11). Many problems of torsional vibration in electrical machinery, diesel motors, and helical shafts can be reduced to this system.

Let J_1, J_2, $J_3 \ldots J_n$ be moments of inertia of the rotating masses with respect to the shaft; ϕ_1, ϕ_2, $\phi_3, \ldots \phi_n$ are the angles of rotation for these masses during vibration, and k_1, k_2, $k_3, \ldots k_n$ are the spring constants of the shaft for lengths ab, bc, cd,...dn, respectively. Then:

$$k_1 (\phi_1 - \phi_2) \quad , \quad k_2 (\phi_2 (\phi_2 - \phi_3)$$

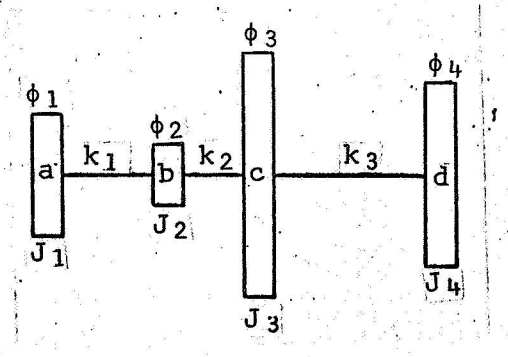


Fig. 11

represent moments of torsion of the above lengths. Before we proceed with the development of the equations of motion, it will be observed that a torsion acts on the first disc during vibration- $k_1 (\phi_1 - \phi_2)$, while for the second disc, the torsion is - $k_1 (\phi_1 - \phi_2) - k_2 (\phi_2 - \phi_3)$; and this is the differential equation of motion for consecutive discs. Applying the same procedure as was used for the system with one degree of freedom, we receive: /38

$$
\begin{aligned}
J_1 \ddot{\phi}_1 + k_1 (\phi_1 - \phi_2) &= 0 \\
J_2 \ddot{\phi}_2 - k_1 (\phi_1 - \phi_2) + k_2 (\phi_2 - \phi_3) &= 0 \\
J_3 \ddot{\phi}_3 - k_2 (\phi_2 - \phi_3) + k_3 (\phi_3 - \phi_4) &= 0 \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots \\
J_{n-1} \phi_{n-1} - k_{n-2} (\phi_{n-2} - \phi_{n-1}) + k_{n-1} (\phi_{n-1} - \phi_n) &= 0 \\
J_n \ddot{\phi}_n - K_{n-1} (\phi_{n-1} - \phi_n) &= 0
\end{aligned}
\qquad (38)
$$

Summing member by member we obtain:

$$J_1 \ddot{\phi}_1 + J_2 \ddot{\phi}_2 + \ldots J_n \ddot{\phi}_n = 0 \qquad (39)$$

which indicates that the moment of motion of the system with respect to the axis of the shaft remains constant during free vibration. In this way, one can continue to exclude any rotation of the shaft as a rigid body and take into consideration only the vibrational motion due to the torsion of the shaft. To find the natural vibrations of the system we proceed by assuming that the solutions of Equations (38) are of the form:

$$\phi_1 = \lambda_1 \cos pt \,, \quad \phi_2 = \lambda_2 \cos pt, \quad \phi_3 = \lambda_3 \cos pt, \ldots$$

$\therefore p = \omega_n$ natural frequency

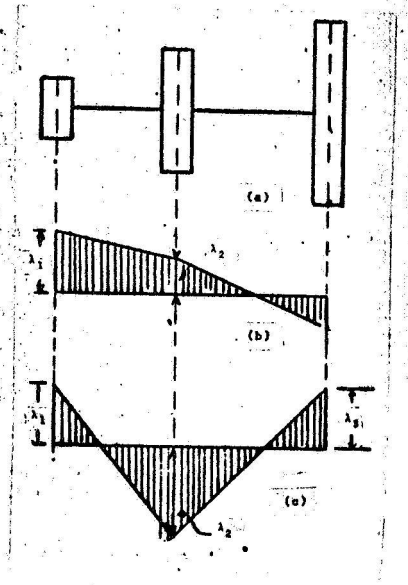
Substituting in Equation (38), we obtain: /39

$$
\begin{aligned}
J_1 \lambda_1 p^2 - k_1 (\lambda_1 - \lambda_2) &= 0 \\
J_2 \lambda_2 p^2 + k_1 (\lambda_1 - \lambda_2) - k_2 (\lambda_2 - \lambda_3) &= 0 \\
&\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots \\
J_n \lambda_n p^2 + k_{n-1} (\lambda_{n-1} - \lambda_n) &= 0
\end{aligned}
\qquad (40)
$$

Eliminating $\lambda_1, \lambda_2 \ldots$ from these equations, we obtain an equation of the n^{th} degree in p^2, which is the "frequency equation".

The n roots of this equation provide us with the n frequencies corresponding to the n principle types of vibration for the system.

We apply the preceding general discussion to the problem of three discs (see Fig. 12). The system of Equation (40) in this case takes the form:



$$\begin{aligned} J_1 \lambda_1 p^2 - k_1 (\lambda_1 - \lambda_2) &= 0 \\ J_2 \lambda_2 p^2 + k_1 (\lambda_1 - \lambda_2) - k_2 (\lambda_2 - \lambda_3) &= 0 \\ J_3 \lambda_3 p^2 + k_2 (\lambda_2 - \lambda_3) &= 0 \end{aligned} \qquad (41)$$ /40

It follows from the first and third of these equations that:

$$\lambda_1 = - \frac{k_1 \lambda_2}{J_1 p^2 - k_1}, \quad \lambda_3 = - \frac{k_2 \lambda_2}{J_3 p^2 - k_2} \qquad (42)$$

Fig. 12

Summing Equation (41) member by member, one obtains:

$$(J_1 \lambda_1 + J_2 \lambda_2 + J_3 + J_3) p^2 = 0$$

and substituting in this Equation (42), we find:

$$p^2 \{ J_1 J_2 J_3 p^4 [(J_1 J_2 + J_1 J_3) k_2 + (J_2 J_3) K_1] p^2 + K_1 K_2 (J_1 + J_2 + J_3) \} = 0$$

This is an equation of the third degree in p^2, one of whose roots is $p^2 = 0$. This root admits of the possibility of making the shaft rotate as a rigid body, without torsion. The other two roots can be rapidly solved by the biquadratic factor:

$$J_1 J_2 J_3 p^4 [k_2 (J_1 J_2 + J_1 J_3) + K_1 (J_2 J_3 + J_1 J_3)] p^2 + K_1 K_2 (J_1 + J_2 + J_3) = 0 \qquad (43)$$

Let p_1^2 and p_2^2 be these two roots. Substituting p_1^2 for p^2 in Equations (42), one finds that:

$$\frac{\lambda_1}{\lambda_2} = - \frac{K_1}{J_1 p_1^2 - K_1}, \quad \frac{\lambda_3}{\lambda_2} = - \frac{K_2}{J_3 p_1^2 - K_2}$$

If p_1^2 is the smallest root, one finds that one of these two /41 expressions is positive, while the other is negative; this indicates that during the vibrations two adjacent discs rotate in one direction,

while the third disc rotates in an opposite direction. This yields the type of vibration shown in Figure 12b; it is assumed that $J_3/K_2 > J_1/K_1$. For the larger root p_2 both expressions are negative, and the type of vibration corresponding to the highest frequency is shown in Figure 12c. During this vibration the disc in the middle rotates in a direction opposite to that of the other two discs.

When the number of the discs is greater than three, the derivation of the frequency equation and its solution become too complicated, for which reason an approximate method is used in these cases. (A method will be developed in Chapter VII).

As with the preceding, it will be noted that for a system of four degrees of freedom the form of the frequency equation is:

$$p^6 - Ap^4 + Bp^2 - C = 0 \qquad A, B, C = \text{Cts.}$$

and the exponents continue to increase according to the increase of the number n for the degrees of freedom; the frequency equation in its general form is then of the form:

$$(p^{n-1})^2 - A(p^{n-2})^2 + B(p^{n-3})^2 - C(p^{n-4})^2 + \ldots = 0$$

2.2.3. Forced Torsional Vibrations in a Shaft.

If one applies a moment of torsion $M_t \sin\omega t$ to one of the discs in Figure 13, forced vibrations are produced, as has already been noted, of period $\tau = 2\Pi/\omega$; in addition, the vibration of each disc will be of the form $\lambda\sin\omega t$. As in the preceding case, we proceed to calculate the equations of motion for the system. Assume that the systems shown in Figure 13 consist of four discs /42
of inertial moments J_1, J_2, J_3, J_4, with spring constants k_1, k_2, k_3. Assume further that the moment of pulsed torsion $m_t = M_t \sin\omega t$ acts on the first disc and that we are looking for the amplitude of the four vibrations for all the discs for a given frequency ω.

The equations of motion for this case are:

$$\begin{aligned} J_1\ddot{\phi}_1 + k_1(\phi_1-\phi_2) &= M_t \sin\omega t \\ J_2\ddot{\phi}_2 - K_1(\phi_1-\phi_2) + k_2(\phi_2-\phi_3) &= 0 \\ J_3\ddot{\phi}_3 - k_2(\phi_2-\phi_3) + k_3(\phi_3-\phi_4) &= 0 \\ J_4\ddot{\phi}_4 - k_3(\phi_3-\phi_4) &= 0 \end{aligned} \qquad (44)$$

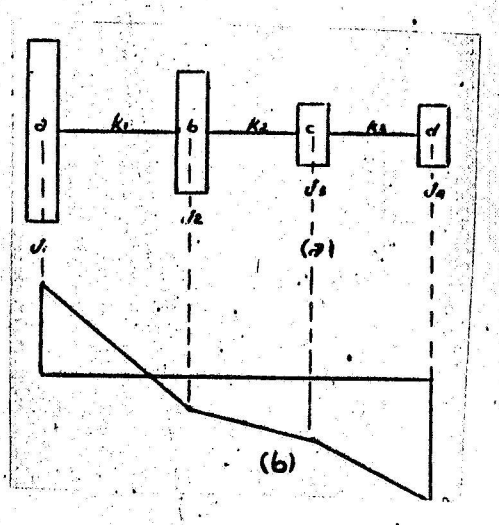


Fig. 13

Substituting in these equations /43 the assumed values for the angles of torsion:

$$\phi_1 = \lambda_1 \sin\omega t \;,\quad \phi_2 = \lambda_2 \sin\omega t, \ldots$$

$$\begin{aligned} J_1\lambda_1\omega^2 - k_1(\lambda_1-\lambda_2) &= -M_t \\ J_2\lambda_2\omega^2 + k_1(\lambda_1-\lambda_2) - k_2(\lambda_2-\lambda_3) &= 0 \\ J_3\lambda_3\omega^2 + k_2(\lambda_2-\lambda_3) - K_3(\lambda_3-\lambda_4) &= 0 \\ J_4\lambda_4\omega^2 + k_3(\lambda_3-\lambda_4) &= 0 \end{aligned} \qquad (45)$$

Summing these equations it is found that:

$$\omega^2(J_1\lambda_1+J_2\lambda_2+J_3\lambda_3+J_4\lambda_4) = -M_t \qquad (46)$$

If λ_1 is the amplitude of the first disc, the amplitude of the second disc is determined by means of the first Equation (45):

$$\lambda_2 = \lambda_1 - \frac{J_1\lambda_1}{K_1}\omega^2 - \frac{Mt}{K_1} \qquad (47)$$

Having obtained this expression for the second of Equations (45), λ_3 is determined and by way of the third of Equations (45) one finds λ_4. Thus, all the amplitudes will be expressed as functions of λ_1. Introducing these into Equation (46), one obtains a lineal equation in λ_1.

It is convenient to place all of the calculations in tabular form as numerical values, as has been done, for instance, in the following table: /44

J	λ	$J_\omega^2\lambda$	$\Sigma J_\omega^2\lambda$	$I/k \Sigma J_\omega^2\lambda$
J_1	I-I	I-2	I-3	8I-4
J_2	2-I	2-2	2-3	2-4
J_3	3-I	3-2	3-3	3-4
J_4	4-1	4-2	4-3	4-4

We begin with the first line of the table. Having found the values for J_1, ω^2 and K_1, we can calculate $J_1\omega^2$ and $J_1\omega^2/K_1$. Beginning with the second line, we calculate λ_2 by means of Equation (47) and the value obtained in the first line. In this way, one obtains the value which appears in the second column, second line. Multiplying this by $\omega^2 J_2$, one obtains the expression in the third column, second line. By summing the expression which occurs in the fourth column, first line and then dividing by K_2, one obtains the two final terms of the second line. When these quantities are obtained, one can begin the third line, using the second of Equations (44) for the calculation of λ_3 and continuing as before. One finally obtains the expression in the fourth line, fourth column which represents the first member of Equation (45). Introducing these expressions into Equation (45), one finds the equation for the calculation of λ_1. The value for λ_1 thus found is substituted in the second column expressions and one obtains values for the vibrational amplitudes of all the discs. Having obtained these amplitudes, we can calculate the torsional angles of the shaft between the consecutive discs which are then equal to $\lambda_1-\lambda_2$, $\lambda_2-\lambda_3$, $\lambda_3-\lambda_4$. With these values for the angles of torsion and with the known dimensions of the shaft, we can determine the force constants produced by the forced vibrations by applying the known formula for the resistance of the materials.

2.4. Torsional Vibrations in Crankshafts of Internal Combustion Engines

/45

Up to this point we have discussed the case of a uniform rod on which were mounted rigid discs. There are, without question,

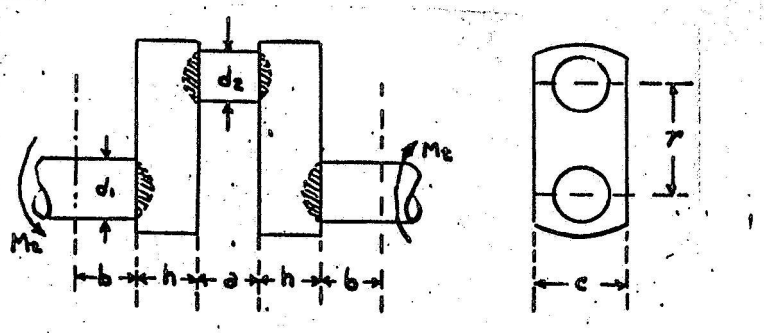


Fig. 14

cases in which the problem of torsional vibration is more complicated. An example of this problem is torsional vibrations in crankshafts. In place of a cylindrical rod, we have here a crankshaft with a complicated form, and in place of the rotating discs, we have crankpins connected to masses with motions reciprocal to the engine. If the crankshaft is replaced by an equivalent cylindrical shaft, one must consider first the torsional rigidity of the crankpin (see Fig. 14). This rigidity depends on the condition of the ligatures in the support. If we suppose that the loopholes in the

supports are such that free displacements are possible during the torsion of the cross sections mn and mn, we can easily obtain the angle of torsion produced for one moment of torsion.

This angle consists of three parts: (a) pivot torsion; (b) crankpin torsion; and (c) flexure of the side of the crankpin.

Let

$C_1 = \pi d_1^4 G/32$ be the torsional rigidity of the pivot,

$C_2 = \pi d_2^4 G/32$ the torsional rigidity of the crankpin,

$B = h_1 c^3 E/12$ the flexional rigidity of the side.

G = shear modulus of elasticity,

E = tension modulus of elasticity.

So that the local deformations of the flank in the faded areas shown in Figure 14 due to torsion can be taken into consideration, the length of the pivot and the crankpin are taken as equal to $2b_1 = 2b + 0.9\ h$ and $a_1 = a + 0.9\ h$, respectively. The crankpin angle of torsion produced for a moment of torsion M_t will then be: /46

$$\theta = \frac{2b_1 M_t}{C_1} + \frac{a_1 M_t}{C_2} + \frac{2r M_t}{B} \qquad (48)$$

In calculating the torsional vibrations of a crankshaft, each crankpin could be replaced by an equivalent shaft of uniform cross section with rigidity C. The length of the equivalent shaft is determined by means of the equation:

$$\frac{M_t l}{C} = \theta \qquad (49)$$

in which θ is the angle of torsion given in Equation (48). Substituting, one obtains:

$$l = C \left(\frac{2b_1}{C_1} + \frac{a_1}{C_2} + \frac{2r}{B} \right) \qquad (50)$$

Another extreme case will be considered (with the supposition that the ligature in the supports is complete) corresponding to zero. In this case the length l of the equivalent shaft will be determined by means of the equation:

$$l = C \left[\frac{2b_1}{C_1} + \frac{a_1}{C_2}\left(1 - \frac{r}{K}\right) + \frac{2r}{B}\left(1 - \frac{r}{2k}\right) \right]. \qquad (51)$$

in which:

$$K = \frac{\frac{r(a+h)^2}{4C_3} + \frac{ar^2}{2C_2} + \frac{a^3}{24B_1} + \frac{r^3}{3B} + \frac{1.2}{G}\left(\frac{a}{2F} + \frac{r}{F_1}\right)}{\frac{ar}{2C_2} + \frac{r^2}{2B}} \qquad (52)$$

$C_3 = c^3 h^3 G/3.6 (c^2+h^2)$ is the torsional rigidity of the side as a rod of rectangular cross section, having sides h and c: h,c $B_1 = \pi d_2^4 E/64$ is the flexional rigidity of the crankpin, and F, F_1 are the cross-sectional areas of the crankpin and the side respectively. /47

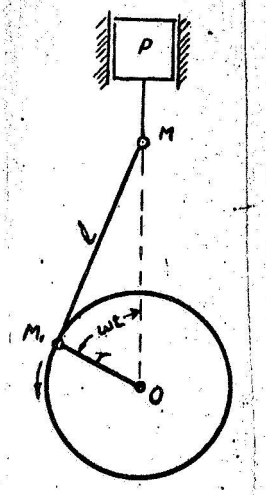


Fig. 15

Taking $a_1 = 2b_1$, $C_1 = C_2$, the whole construction, as is shown in Equations (50) and (51), reduces the entire length of the shaft to a ratio of $1/[1 - (r/2k)]$. Under real conditions, the length of the equivalent shaft will have a value lying between the two extreme cases previously considered.

Another problem which has to be resolved in the study of torsional vibration of crankshafts is the calculation of inertia for masses in motion. Let us suppose that the mass m of the connecting rod is replaced by two masses, $m_1 = (J/l^2)\pi$ in the pin portion of the crankpin and $m_2 = m - (J/l^2)$ in the cross piece of the piston, where J denotes the moment of inertia of the connecting rod with respect to the center of the cross piece. All of the moving masses can be replaced by masses concentrated in the same two points, in such a way that, finally, it is only necessary to take into consideration two masses M and M_1 (Figure 15).

Let ω be a constant angular velocity; ωt the angle of the crankpin measured to the dead point, as is shown in Figure 15. Then /48 the velocity of mass M_1 is equal to ωr, and the velocity of the mass M, as is easy to see, is equal to:

$$\omega r \sin\omega t + \frac{r^2\omega}{2l} \sin 2\omega t$$

The kinetic energy of the moving masses of a crankpin is:

$$T = \frac{1}{2} M_1 \omega^2 r^2 + \frac{1}{2} M\omega^2 r^2 \left(\sin\omega t + \frac{r}{2l}\sin 2\omega t\right)^2$$

The average value of T during one revolution is:

$$T_m = \frac{1}{2\pi} \int_0^{2\pi} T d(\omega t) = \frac{1}{2} \left[M_1 + \frac{1}{2} M \left(1 + \frac{r^2}{4l^2} \right) \right] \omega^2 r^2$$

By employing this average value, the inertia of the moving part connected to a crankpin can be replaced by the inertia of an equivalent disc whose moment of inertia is:

$$J = \left[M_1 + \frac{1}{2} M \left(1 + \frac{r^2}{4l^2} \right) \right] r^2$$

If we replace all of the crankpins by shafts of equivalent length, the problem of crankshaft vibration is reduced to the torsional vibration of a cylindrical shaft and the critical velocities can be calculated more easily.

It must be noted that in this system the forced vibrations are not caused solely by the pressure on the piston due to expanding gases, but also by the incomplete equilibrium of the parts having reciprocal motion. Practically all the phenomena associated with /49
dangerous critical velocities would develop if one cut off the flow of fuel and made the motor operate without resistance at the required velocity.

There exist various means of resolution for design of systems subject to vibrations which deal with the simplification of the study and make the system more representative and approximate somewhat more closely the real conditions operating in the systems under discussion. In the following chapters, several ideas will be put forward which will bring us to a better understanding of design.

CHAPTER III

MODELS

3.1. Introduction

As was mentioned in the preceding chapter, it is not always /50
convenient to solve vibrational problems by purely mathematical methods since the work involved is quite laborious and the accuracy of the mathematical method is not very different from that which is obtained with approximate methods which are quite acceptable for purposes of design and control.

A method which is sufficiently exact and one which gives us an idea of a system equal to the real system is the method of scale models. The model need not be constructed on an exact scale nor need the scale model have exactly the same form as the real system.

A model of a prototype mechanical system is defined, then, as a model designed and tested under the operating conditions of a real system within a certain range of exactness (the mechanical prototype is the real mechanical system).

3.2. Types of Models

The types of models which can be considered mechanical prototypes are:

(a) A geometrically exact scale model usually smaller in size /51
than the prototype; however, in special cases the model can be constructed of the same size or larger than the prototype.

(b) A mechanical model used under special conditions does not require exact geometric scale. Such scalar differences can be justified by: (1) the small influence in operation of the model on the prototype; (2) a fixed error acceptable in the prototype operation; (3) the possibility of correcting in the calculation for differences between the scale model and the prototype.

(c) A model consisting of a completely dissimilar system, such as an electrical model. The modeling conditions require that the electrical model satisfy the same mathematical relations as the mechanical prototype (see Chapter IV).

(d) A model computer system which contains all the elements and which is designed to carry out the mathematical operations indicated by the equations of motion of the prototype. A differential analyzer can be considered a model.

(e) Various combinations of the above-mentioned types.

The method which will be studied in this chapter is of type (a), and its principal advantages are:

(1) When the prototype has very large physical dimensions, the small scale model is inexpensive to construct and to test.

(2) Models can often be designed in a form such that measurements in the laboratory can be as simple as those in the prototype. This means also that repeated tests can be carried out with advantages for the researcher.

(3) One can design models which are dynamically equal to the prototype and which have a simpler form than that of the prototype.

And as with all other things, the method also has its disadvantages: /52

(1) It is sometimes impossible to design models dynamically equal to the prototype;

(2) It is rarely possible to reproduce a small scale model with all the structural details of the prototype;

(3) The exactness required in the measurements of small models is difficult to attain.

3.3. Conditions for Testing and Design of Models

The conditions for designing and testing a model can be obtained in several different ways which, however, are based on the same fundamental principle; there are, however, advantages in considering them separately:

(1) If the basic mechanical quantities of force, mass, length, and time are known, these are sufficient to describe the operation of the prototype; the laws for the model can then be based on the general principle of dynamic similarity. The model conditions in such a case are comparatively restrictive and studies on models of complex systems can be very difficult.

(2) If one has at one's disposal a large amount of information about the problem, one can eliminate from the known quantities those which are of no importance. The model conditions based on a dimensional analysis are less restrictive, and the modeling techniques can be simplified.

(3) If, in addition to the variables involved, the equations of motion which describe the prototype are known - in the form of differential or integral equations - an ample number of the additional parameters of the model can be found by reducing their complexity.

All of the methods for establishing conditions of design and testing of models can be derived from the basic principles of dimensional analysis.

3.4. Dimensional Analysis

3.4.1. Definitions.

A physical quantity can be measured by comparing it with a specimen of a known quantity from the same quantity. This known quantity is called "unit". For the complete definition of any physical quantity, both the unit which is employed and the number of units contained in the sample must be given. /53

The physical quantities which enter into a mechanical problem are commonly expressed in terms of three fundamental units chosen arbitrarily. If these fundamental units are mass, length, and time, the system of units is known as an absolute system. If the units are force, length, and time, the system of units is termed a gravitational system. In the English gravitational system, the unit "foot" is used to designate length; for force, the unit is "pound force"; for time, the unit is the "second". The metric gravitational system employs the meter, kilogram (force), and second. The cgs absolute system uses centimeter, gram (mass), and second; and the mks absolute system uses the meter, kilogram (mass), and second.

The three fundamental units can be combined conveniently to obtain the units for other mechanical quantities. Units thus obtained are known as "derived units".

3.4.2. Dimensions.

The quantities expressed in derived units are known as "secondary quantities" in order to distinguish them from the primary quantities which are measured in basic units. In general, a secondary quantity Q_s takes the form:

$$Q_s = (n_1 a)^{\alpha} (n_2 b)^{\beta} (n_3 c)^{\tau} \quad [F^{\alpha} L^{\beta} T^{\tau}] \tag{53}$$

where n_1, n_2, and n_3 represent the number of units contained in a given quantity; and a, b, and c are the numbers which indicate the size of the conversions in the unit for the number of units contained in the conversion of the given sample. The bracketed

capital letters do not represent quantities; they indicate only the fundamental units.

It is evident from Equation (53) that the expression for the two measured values of a secondary quantity is independent of the size of the units included, since the constants a, b, and c are cancelled. /54

The exponents α, β, and τ are, by definition, the dimensions of the derived unit $[F^{\alpha}L^{\beta}T^{\tau}]$. The dimensions of some secondary quantities are given in Table 2.

TABLE 2. DIMENSIONS OF COMMON MECHANICAL QUANTITIES

Quantity	Dimensional Form	
	Gravitational System	Absolute System
Force	F	MLT^{-2}
Mass	FT^2L^{-1}	M
Velocity	LT^{-1}	LT^{-1}
Acceleration	LT^{-2}	LT^{-2}
Area	L^2	L^2
Volume	L^3	L^3
Density	FT^2L^{-4}	ML^{-3}
Momentum	FT	MLT^{-1}
Work, Energy, Moment	FL	ML^2T^{-2}
Potential	FLT^{-1}	ML^2T^{-3}
Pressure, Force	FL^{-2}	$ML^{-1}T^{-2}$
Viscosity	$FL^{-2}T$	$ML^{-1}T^{-1}$
Angle	Without dimension	Without dimension

It can be shown, in general terms, that if the secondary quantities are independent of the size of the basic units, the secondary quantities should take the form of power products of the primary quantities. It follows that the quantities encountered in mechanics always take the forms shown in Table 2. /55

3.4.3. Dimensional Homogeneity.

The equations which describe the physical processes must be dimensionally homogeneous; that is to say, the dimensions on the left-hand side of the equations should be equal to the dimensions on the right-hand side. For example, let us consider the displacement equation of a simple mass spring system:

$$x = A\sin\sqrt{\frac{k}{m}}\,t$$

Dimensionally

$$x = [L] \qquad A = [L]$$

and

$$\sqrt{\frac{k}{m}}\,t = \left[\sqrt{\frac{F/L}{FT^2L^{-1}}}\,T\right] = I$$

Then since [L] = [L], the equation is dimensionally correct.

A dimensionally homogeneous equation can always be stated in the form:

$$\Pi_1 = \phi\,(\Pi_2, \Pi_3 \ldots)$$

where Π_1, Π_2, Π_3,... are dimensionless factors and ϕ indicates any functional relation.

For example, $x = A\sin\sqrt{\frac{k}{m}}\,t$ can be written:

$$\left(\frac{x}{A}\right) = \phi\left(\sqrt{\frac{k}{m}}\,t\right)$$

where the quantities $(\frac{x}{A})$ and $(\sqrt{\frac{k}{m}}\,t)$ are dimensionless quantities. /56

3.4.4. Dimensional Analysis.

A consideration of the dimensions of the quantities which arise in a physical problem can yield useful information with regard to the nature of the solution. The equations included in the basic laws of mechanics are independent of the size of the basic units, and any equation derived from these basic laws also has the same independence. Such equations are called "physical equations."

Let us assume that in a particular system containing N basic units, there are n variables $(\nu_1, \nu_2, \ldots \nu_n)$ which are important. The operation of the system is described by a physical equation:

$$f\,(\nu_1, \nu_2, \ldots, \nu_n) = 0$$

If it is required that the function $f(\nu_1, \nu_2, \ldots, \nu_n)$ remain unaltered during a change in the size of the basic units which include ν, it is always possible to state the equation in the form:

$$\phi(\Pi_1, \Pi_2 \ldots, \Pi_{n-N}) = 0$$

where the terms Π are dimensionless factors. The factors Π are composed of power products of the variables ν and include no more than (N + 1) variables in any factor. Since ν are formed from power products of the basic units, the dimensionless factors Π should be power products of the basic units.

These considerations can be summarized by the Buckingham potential theorem.

A physical equation having n variables in N basic units can be written as follows: /57

$$\phi\ (\ \Pi_1, \Pi_2\ , \ldots \Pi_{n-N}\ ) = 0 \qquad (54)$$

where the factors Π are independent dimensionless terms having the form of power products of the variables. The number of Π terms will be not less than (n-N).

3.5. Theory of Models

The basic theory of models can be developed directly from the preceding discussion of dimensional analysis. Let us assume that the important variables in the system are known; Equation (54) shows that the variables can be grouped in dimensionless groups so that the equation which describes the system will be

$$\phi\ (\Pi_1\ ,\ \Pi_2\ ,\ \ldots) = 0$$

Since these Π terms are independent, each one will contain one variable which does not appear in the other terms. We solve this equation for Π which contains the variable of primary importance in the particular problem:

$$\Pi_1 \ = \ \theta\ (\Pi_2\ ,\Pi_3,\ \ldots)$$

Since this is a general equation, it will apply to the prototype as well as the model:

$$\Pi_1\ \rho, = \ \theta\ (\ \Pi_2\ \rho,\ \Pi_3\ \rho,\ \ldots)$$

$$\Pi_1\ m \ = \ \theta\ (\ \Pi_2\ m,\ \ \Pi_3\ m,\ \ldots)$$

If the model is designed and tested so that:

$$\Pi_2\ m \ = \ \Pi_2\ \rho\ ,\ \Pi_3\ \ m\ =\ \Pi_3\ \rho, \qquad (55)$$

then the functions are equal: /58

$$\theta\ (\ \Pi_2\ \rho,\ \ \Pi_3\ \rho,\ \ldots) = \theta\ (\Pi_2 m,\ \Pi_3\ m\ \ \ldots)$$

Then:

$$\Pi_1 \rho \ = \ \Pi_1\ m \qquad (56)$$

Equations (55) are the conditions specified in design, and methods of testing the models and Equation (56) show that the operation of the prototype can be defined in terms of the model in operation. Equations (55) are known as "equations of design" and Equation (56) is called a "predictor."

3.5.1. Principle of Dynamic Similarity

If the basic quantities force, mass, length, and time are known, the general theory of models discussed in the previous section leads directly to the basic relations which should always exist between model and prototype. Following the procedure outlined in the preceding section, the variables involved are force F, mass m, length l, and time t. From these four variables one can form a factor Π, which can be:

$$\phi \left(\frac{F t^2}{ml} \right) = 0$$

Since [by Equation (56)] this factor is the same for the model and the prototype, the dynamic similarity is obtained completely from the equation:

$$\frac{F_m t_m^2}{m_m l_m} = \frac{F_p t_p^2}{m_p L_p}$$

This equation is determined by such scale factors as the following:

$$\lambda_F = \frac{F_p}{F_m}, \quad \lambda_t = \frac{t_p}{t_m}, \quad \lambda_m = \frac{m_p}{m_m}, \quad \lambda_l = \frac{l_p}{l_m}$$

/59

The relationship between the model and the prototype is then:

$$\lambda_F = \frac{\lambda_l \lambda_m}{\lambda_t^2} \qquad (57)$$

This equation is an analytical expression of the principle of dynamic similarity.

3.5.2. Analysis of the Model Based on the Equations of Motion.

The previous section assumed that only a limited quantity of information was known (F, m, l, and t) concerning the problem. If one obtains additional information, the conditions can change

considerably, and in many cases, the specific limitations of the problem decrease. This additional information can often be had from the equation of motion of the system which it describes. In this way, much useful information concerning the factors in the scale model can be obtained, even though the solution of the equations is unknown. In addition, this provides a link between the prototype and the model, since the equations of motion apply to the prototype as well as the model.

There are two different ways in which a system of equations can be used to develop the conditions of the model. The first method assists in finding, from the equation, groups of parameters which can be utilized in the same way as Π in the discussion in the preceding section. These parameters, in reality, are combinations of Π factors, with the advantage that the total number of independent parameters for the problem is reduced. The second method involves a transformation of all the quantities in the equation by means of an ensemble of dimensionless parameters which yields a system independent of size, and which can be applied directly to the prototype and the model. /60

3.6. Construction and Testing of the Model

The practical problems of constructing and testing of models, according to the above-mentioned theory, can be grouped as follows: materials, construction techniques, methods of charging and measuring the operation of the model. We will discuss these topics in the following section.

3.6.1. Materials.

The materials of which the scale models are constructed must not only have physical properties that satisfy the basic requirements of modeling, but must also take into account other factors, such as ease of manufacture, ease of measuring the operation of the model, roughness, stability during changes of temperature and humidity, etc. Small scale models often contain delicate components or other physically small structural elements, which with some materials, results in difficulties in manufacture or in handling.

Table 3 shows several materials and their properties which are used in the construction of scale models.

The expression (E/ρ) occurs in many design requirements for models regarding problems of vibration. This expression is not at all constant for the common metals. For models with a large scale factor in which the factor (E/ρ) is greatly reduced, it is necessary to employ plastics or materials similar to oilcloth; however, such materials can have undesirable properties, such as lack of stability and linearity, and high internal friction.

3.6.2. Construction Techniques.

The problems in construction of models depend upon the material being used, the size of the model and the number of details which are required in it. A geometrically exact reproduction of the prototype is not always convenient, due to the fact that the accuracy /61 which can be obtained by such a reproduction is not much greater than that which can be obtained from a model constructed more simply according to the principle of dynamic similarity. However, the ingenuity of the person creating the model enters in at this point. These cases require a careful program of static and dynamic tests in order to compare the real properties of the model.

One of the principal disadvantages encountered in the construction of models is the fact that it is very difficult to effect an agreement between the specific damping of the prototype and that of the model. This is due to the coupling joints in addition to the internal damping of the material with which the model is to be constructed.

3.6.3. Methods of Loading.

The loading tests which are applied to the method are of three types: static, periodic, and transitory.

Static charge presents no problems in the testing of the model; the dynamic loading, however, presents difficulty as to the appropriate form for applying it. For periodic loads, mechanical or magnetic force generators can be used for their application. However, when the model is small in size, difficulties arise in the coupling of models and generator, and in this case the model-generator group should be considered one system.

The application of a transitory load can be accomplished by the sudden loosening of the load which is a transient load of the impulse function type. This can be accomplished by applying a load across a spring or wire which can be quickly shortened or can contain a link which is able to expand rapidly.

The design conditions for the model require, on occasion, large amounts of load that can produce difficulties in the model or lead to inexact measurements. For these cases we can adopt distorted scales or select a different type of model. An important /62 part of the loading problem is the measurement of the load; a great variety of measuring devices and transducers exist for this purpose.

Table 3 shows several materials and their properties which are used in the construction of scale models.

TABLE 3. PHYSICAL PROPERTIES OF MATERIALS USED IN THE CONSTRUCTION OF MODELS

Material	Specific Weight lb/ft^3	E Static Modulus of Elasticity lb/in^2	$\frac{E}{\rho}$, $\frac{ft^2}{sec}$	Limit of Elasticity lb/in^2	Rupture Stress lb/in^2	Logarithmic Damping Decrement
Steel	490	30×10^6	2.84×10^8	60 000	80 000	0.005
Duraluminum	170	10.3×10^6	2.81	50 000	70 000	0.002
Magnesium Alloy	110	6.5×10^6	2.74	30 000	40 000	
Titanium Alloy	280	15×10^6	2.48	75 000	100 000	
Bronze	540	14×10^6	1.20	50 000	80 000	0.005
Concrete	150	3.3×10^6	1.02	...	4 000	0.075
White Oak	50	1.7×10^6	1.53	5 000	8 500	0.03
Celluloid	90	0.15 a 0.40×10^6	0.08	6 000	7 000	0.28 a 0.35
Plexiglas (Lucite)	74	0.40 a 0.60×10^6	0.25	8 000	9 000	0.16 a 0.24
Tenite II	74	0.06 a 0.35×10^6	0.04	1 200 a 6 100	1 900 a 6 800	0.16 a 0.22
Oilcloth (Hardness 30)	60	130		1 000	3 000	0.1
Oilcloth (Hardness 30)	60	370		1 000	3 000	0.5

CHAPTER IV

ANALOGIES

4.1. Introduction

It has been noted in the preceding chapter that the model can /64
be completely different from the prototype, as is the case with an electrical model. This can be if we know the differential equations of motion which describe the system and compare them with those which result from an electrical circuit, in such a way that both have the same form. In this case it can be stated that both systems operate equally since the solution of the differential equations is the same for both, with the exception that the values of the constants which is different for both systems is converted; or converting the coefficient and the variable depending on the model, one obtains the solution of the system under study. These systems are known as "analogous systems" or "electromechanical analogies."

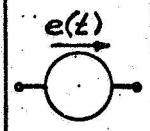
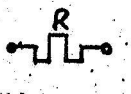
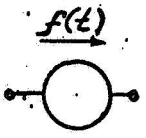
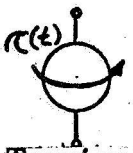
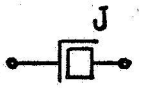
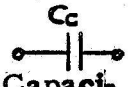
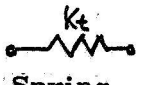
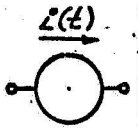
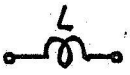
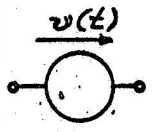
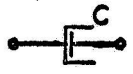
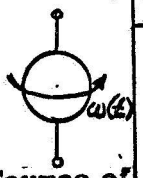
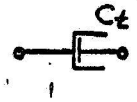
One of the principal advantages which is obtained with this type of system is that it is very easy to work with electrical circuits, since in these circuits it is simpler to measure and control current voltages and frequencies, and by appropriate circuit design, it is possible to obtain quantitative results. It is also possible to find the response spectrum for a broad range of frequencies, because it is easier to vary the excitation frequency. /65

Two distinct types of electrical analogies are possible; in electrical terminology they are known as duals of each other. That is, they are governed by the same sets of differential equations. The first is a force-voltage analogy and is based on the second Kirchhoff law (In any network the algebraic sum of all the current flowing at a node is equal to zero).

The elements of an electrical circuit are termed active and passive; the active elements are those generated by voltage and current; the passive are the resistances, inductances, and capacitances. Similarly, in a mechanical system, the active elements to be considered are the exciters of force and motion; the passive elements are dampers, masses, and springs.

These elements and the conventional symbols which represent them are given in Table 4.

TABLE 4. SYMBOLS /66

Electrical System		Mechanical System			
		Rectilinear or Translational		Rotational	
Active	Passive	Active	Passive	Active	Passive
 Voltage source	 Resistance	 Origin of force	Mass	 Torsion source	 Mass
 Capacitance	Spring	 Spring			
 Current source	 Inductance	 Velocity source	 Damper	 Source of angular velocity	 Damper

It can be observed from the demonstration in Chapter II that, by the definition of dual, the rectilineal or translational vibration equations are duals of the rotational vibration equations since, as can be seen, the form of the equation is the same.

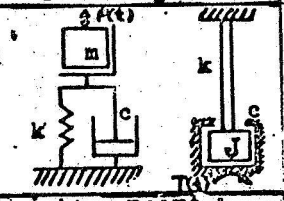
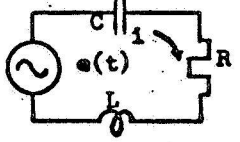
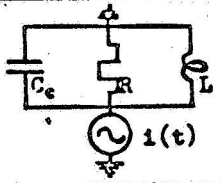
Through electrical circuits we can derive the fact that a series circuit and a parallel circuit have exactly the same equation, the first on the basis of voltages (loops) and the other on the basis of currents (nodes). We observe that the sources of current are the duals of the sources of voltage; the conductance is the dual of the resistance; the capacitance is the dual of the inductance; /67 and the inverse inductance is the dual of the elasticity. This can be shown in a relatively simple manner as seen in Table 5.

4.2. Procedure for Construction of a Symbolic Diagram

The procedure generally employed for finding the electrical analogy of the mechanical system under study consists, first, of reducing the real mechanical system to a symbolic diagram using the conventional symbols given in Table 4. Having developed this symbolic diagram, one proceeds, by comparison and study, to transform it into an electrical diagram; this is quite simple when one knows the equations for the electrical networks; however, to

facilitate even more the analogous transformation of the symbolic diagram, one makes use of simple tables, which directly provide the analogies betwiin mechanical and elecrical systems. These systems, as is shown in Table 5.

TABLE 5. ELECTROMECHANICAL ANALOGIES /68

Mechanical System Rectilineal Torsional	Equivalent Electrical Systems	
	Force-Voltage Analogies	Force-Current Analogies
Force, Torque f,t	Voltage e	Current i
Damping C,C_t	Resistance R	Conductance I/R
Mass, Inertia m,J I/k,I/k	Inductance L	Capacitance C_c
Velocity v,θ	Capacitance C	Inductance L
Displacement x,θ	Current i	Voltage e $\int e dt$
	Load q	
Damping $f = cv = c\dot{x}$	R Resistance $e = Ri = R\dot{q}$	R Resistance $i = e/R;\ q = \dot{e}/R$
Mass $f = m\dot{v} = m\ddot{x}$	L Inductance $e = Ldi/dt = L\ddot{q}$	C_c Capacitance $i = C\dot{e};\ q = C\ddot{e}$
Spring $f = kx = k\int v dt$	C_c Capacitance $e = q/C\ (\int i dt)/C$	L Inductance $i = (\int e dt)/L;\ q = e/L$
 	 	
At a point: $\Sigma f = 0\ ;\ \Sigma t = 0$	For the loop: $\Sigma(\Delta e) = 0$	At a point: $\Sigma i = 0$
$m\dot{v} + Cv + k\int v dt = F(t)$ $m\ddot{x} + C\dot{x} + kx = F(t)$	$L\frac{di}{dt} + Ri + \frac{I}{C_c}\int i dt = e(t)$ $L\ddot{q} + R\dot{q} + \frac{q}{C_c} = e(t)$	$C\dot{e} + \frac{e}{R} + \frac{I}{L}\int e dt = i(t)$ $C\ddot{e} + \frac{I}{R}\dot{e} + \frac{e}{L} = \frac{di(t)}{dt}$

The procedure followed to find the symbolic diagram for a mechanical system is as follows: /69

(a) Identify the two terminals of each active element and each passive element.

(b) Connect, at a common junction, those teams which move together.

(c) Connect to each reference junction all the terminals which remain stationary with respect to the reference structure.

(d) Mark each active element (source) with an arrow to indicate the positive direction of the force or velocity which it represents.

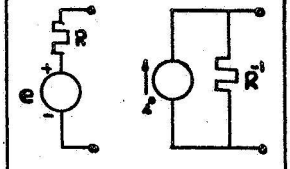
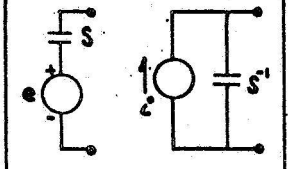
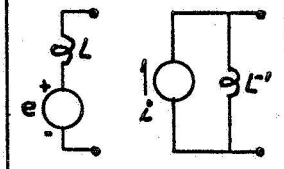
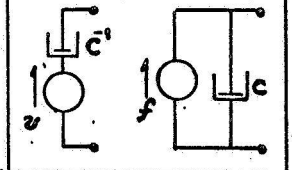
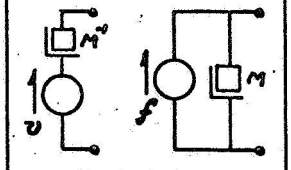
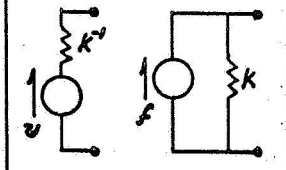
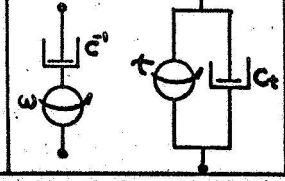
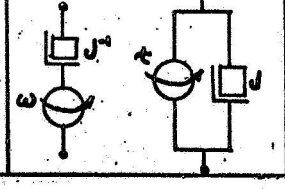
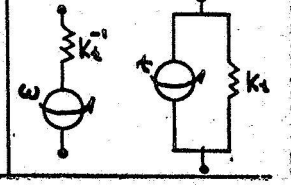
(e) Assign a coordinate to each movable junction and indicate with an arrow the direction for positive values of this coordinate.

4.3. Exchanges of Sources

The exchange of excitation sources is a useful tool in establishing the analogies between mechanical and electrical systems, since with this method, sources of voltage can be substituted for sources of current in electrical systems; and in mechanical systems, sources of velocity can be substituted for sources of force. The advantage of making these exchanges is that the procedure makes it easier to establish the differential equations, since for example, the differential equations which are obtained for a circuit on the basis of nodes are equations of current, and if sources of voltage exist in this circuit it is convenient to have only sources of current. A similar exchange can be made when one formulates equations on the basis of voltages (loops).

The possible exchanges are grouped in Table 6; these exchanges can be made in electrical networks, in mechanical systems (in rectilineal as well as torsional mechanical systems). /70

TABLE 6. EXCHANGE OF SOURCES, MAINTAINING THE TERMINAL CONDITIONS WITHOUT VARIANCE

Electrical Networks			
Translational Mechanical Networks			
Rotational Mechanical Networks			

The proof of these exchanges is purely analytical so that to effect these exchanges, the initial and final conditions remain invariable.

The mechanical networks shown in Figure 18 can be transformed bearing Table 6 and Equations (48) in mind--by substituting the source of velocity for the source of force in the following manner. The source of velocity $v_3(t)$ in series with the parallel group consisting of C_2 and K_2 is replaced by two sources of force, which added together yield a single source of force $f_2(t)$ in parallel with C_2 and K_2. The new mechanical network obtained from the network of Figure 18 is shown in Figure 19.

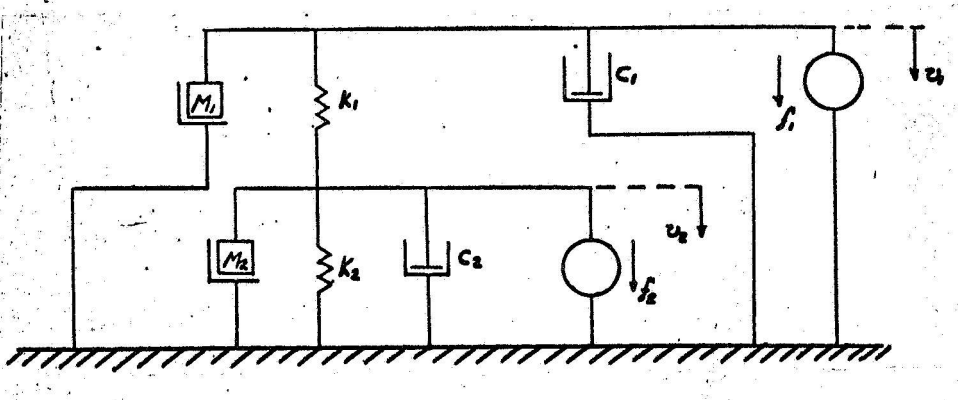


Fig. 19 /71

The initial conditions for the figures shown in Figures 18 and 19 are:

$$x_1(0) = a_1 \qquad \dot{x}_1(0) = -b_1$$

$$x_2(0) = -a_2 \qquad \dot{x}_2(0) = b_2$$

since we choose the initial equilibrium position of the mass as the origin of the coordinates.

4.4. Systems with One Degree of Freedom

By way of illustration, we shall proceed to find the symbolic diagram for a system of one degree of freedom as shown in Figure 1 of Chapter II.

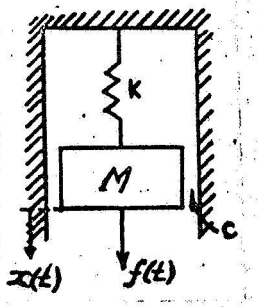


Fig. 1 (repeated)

At t = 0, the body is at rest and is in its equilibrium position or a position in which the spring is extended sufficiently to support the weight Mg (g = acceleration due to gravity). The /72

support is considered fixed and the specific mass of the spring is negligible in comparison with M. Since M can have only one rectilineal vertical motion, one coordinate only is sufficient to describe the position of M at any instant. Following the convention established in Chapter II, which states that the equation of motion is [see Equation (1)], one obtains:

$$M\ddot{x}+C\dot{x}+x = f(t) \qquad M\frac{d^2x}{dt^2} + C\frac{dx}{d_t} + Kx = f(t) \qquad (I)$$

$$f(t) = \mathrm{Sin}\omega t$$

And the symbolic diagram which corresponds to Figure 1 - making use of Table 4 and following the above procedure - is:

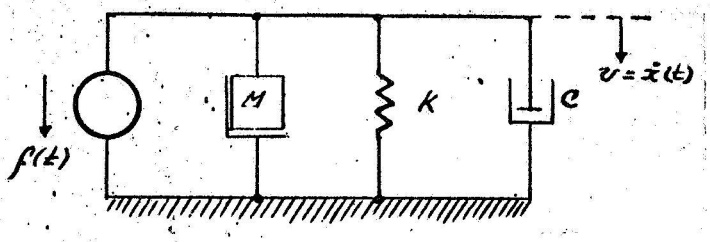


Fig. 16. Symbolic Diagram of the System Shown in Figure 1.

in which the corresponding initial conditions are: for t = 0 and x(0) = 0 and $\dot{x}$ (0) = 0, which corresponds to the position of equilibrium for M. /73

4.5. Systems with Two Degrees of Freedom

The mass M_1 of Figure 17 is supported by a spring above the mass M_2 and this is supported elastically above a vertically moving structure with displacement $x_3(t)$ which is known with relation to a fixed reference; motion downward is taken as positive.

Both are forced by guides to move only in the vertical directions--a viscous damping C_1 and C_2 between the masses M_1 and M_2 and the guides, respectively. The springs have a rigidity K_1 and K_2 as is indicated in Figure 18.

The masses have initial displacement and velocities from the equilibrium positions as follows:

Mass	Displacement	Velocity
M_1	a_1 units downward	b_1 units upward
M_2	a_2 units upward	b_2 units downward

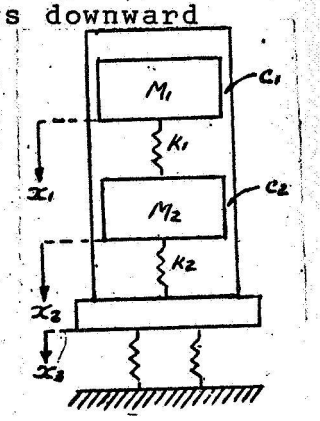


Fig. 17

The symbolic diagram corresponding to Figure 17 (following the procedure given previously) is shown in Figure 18. /174

Since the velocity of the structure is known, we will employ a velocity source $v_3(t) \triangleq \dot{x}_3(t)$ as a source of excitation.

To find the equations of motion for this system we will apply Newton's first law first to the mass M_1 and then to the mass M_2; one obtains:

$$M_1 \frac{d^2 x_1}{dt^2} = K_1 (x_2 - x_1) + C_1 \frac{d}{dt} (x_3 - x_1)$$

$$M_2 \frac{d^2 x_2}{dt^2} = - K_1 (x_2 - x_1) + K_2 (x_3 - x_2) + C_2 \frac{d}{dt} (x_3 - x_2) \qquad (58)$$

in which $x_1 \triangleq x_1(t)$, $x_2 \triangleq x_2(t)$ and $x_3 \triangleq x_3(t)$. The signs of Equa-

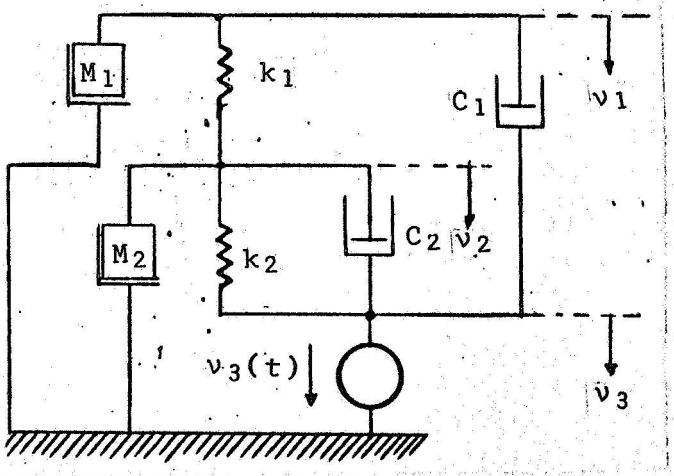


Fig. 18

tions (47) are justified according to the following reasoning: If /75
x_2 is greater than x_1, the spring K_1 is extended by dragging downward to M_1; this tends to give M_1 a positive acceleration. In other words, the extension of K_1 drags M_2 upward and tends to give M_2 a negative acceleration. If x_3 is greater than x_1 the guide is moved downward more quickly than M_1. This drags M_1 downward and tends to give it a positive acceleration. Similarly, M_2 is dragged downward if x_3 is greater than x_2.

Equations (47) can be stated in the symmetrical form:

$$\begin{aligned} M_1\ddot{x}_1 + C_1\dot{x}_1 + K_1x_1 - K_1x_2 &= C_1\dot{x}_3 \triangleq f_1(t) \\ -K_1x_1 + M_2\ddot{x}_2 + C_2\dot{x}_2 + (K_1 + K_2)x_2 &= C_2\dot{x}_3 + K_2x_3 \triangleq f_2(t) \end{aligned} \qquad (59)$$

Since $x_3(t)$ and its derivative $\dot{x}_3(t)$ are known functions of time, $f_1(t) \triangleq C_1\dot{x}_3(t)$ is a force applied to a known M_1.

Similarly, $f_2(t) = C_2\dot{x}_3(t) + K_2\int\dot{x}_3(t)\,dt$ is a known force applied to M_2.

4.6. Formulation of Electrical Analyses

The analogous systems have been defined in Section 4.1 as different physical representations of the same set of differential equations (in another case as general integral differential equations). The electrical analyses for a mechanical system can be constructed by finding an electrical network which has a set of integral differential equations of the same form as the set for the mechanical system. In many cases, the analogy can be found intuitively, but it is always better to obtain a correct analogy by pursuing the following steps:

(1) Describe the equations for the prototype system.
(2) Rewrite the equations, using the constants and variables for the electrical network.
(3) Interpret these equations for the electrical network by tracing the network whose operation they describe.

As was mentioned in Section 4.1, within certain limits, the /76
electrical networks have duals. There is, then, more than one possibility for electrical analogy of a mechanical system. By way of illustration, let us consider the system with one degree of freedom shown in Figure 1. The lineal equation with constant coefficients [Equation (1)] for this mechanical system, derived through the application of Newton's second law, is a force equation. For the electrical system the equation of the same type derived through the application of Kirchhoff's laws are equations either of currents or voltages. It is then possible to have i∿f (read: current analogous to force) or to have e∿f (voltage analogous to force). The two networks which can thus be obtained will be duals since the voltages in the node pairs in one will be analogous to the loop current in the other.

Equation (1) for the system of Figure 1 (following the above procedure) can be rewritten as a function of the velocity $v \triangleq dx/dt$ in order to facilitate the comparison:

$$M \frac{dv}{dt} + Cv + K \int v\,dt = f(t) \qquad (60)$$

Making i∿f, the analogous equation of current is:

$$C_c \frac{de}{dt} + Ge + \Gamma \int e\,dt = i(t)$$

C_c = Capacitance
G = I/R Conductance
Γ = 1/L Inverse inductance

This equation corresponds to the network shown in Figure 20 (network with a pair of nodes).

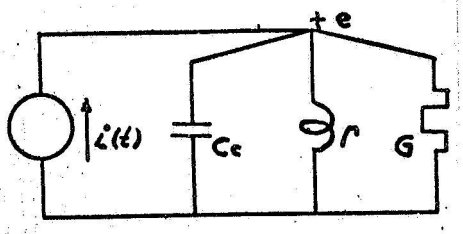


Fig. 20.

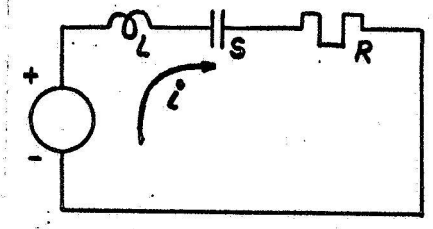


Fig. 21.

/77

This network has the same geometric or topological model as the mechanical network or symbolic diagram shown in Figure 16. Therefore, the electrical analogy can be constructed directly from the symbolic diagram if one makes the following substitutions:

Source of current	i (t)	for	f (t) source of force
Passive elements	C_c	for	M
	G	for	C
	Γ	for	K
Unknown	e (t)	for	v (t)

Making e∿f, the equation for the voltage drops will be analogous to Equation (60), or let: $L \frac{di}{dt} + Ri + S \int i\,dt = e(t)$ (62) $S = \frac{1}{C_c}$ elasticity, which corresponds to the network of Figure 21 with a loop. Topologically, this electrical network is the dual of the symbolic diagram (mechanical network) shown in Figure 16 and also of the electrical network given in Figure 20.

The network of Figure 21 can be directly constructed on the

basis of this topological duality by making:
(1) the topological dual of the mechanical network and
(2) substituting the source of velocity e(t) for the source of voltage v(t)

/78

Passive elements	L	for K^{-1}
	R	for C^{-1}
	S	for M^{-1}
Unknown	i (t)	for f (t)

If Equation (61) is derived, one obtains:

$$C_c \frac{d^2e}{dt^2} + G \frac{de}{dt} + \Gamma e = \frac{di}{dt} \qquad (63)$$

This equation, derived from the node current, has essentially the same form as Equation (60) when this is applied as a function of displacement x instead of velocity v or, alternately, Equation (1).

The network which corresponds to Equation (63) differs from that shown in Figure 21 in that, instead of having current, we have its derivative. In the representation of the analogous network one will have:

$$\frac{di}{dt} \sim f \quad , \quad i \sim \int f dt \quad y \quad e \sim x$$

Three analogous networks have been developed for the mechanical system in Figure 1. These can be developed further in the same way by repeating the process of integration or derivation starting from Equations (61) and (62). A summary of the constants, variables, and analogous geometric forms of network diagrams for mechanical and electrical systems for four bases of analogies is given in Table 7.

One can establish other bases by exchanging the column of mechanical variables, as for example, if one makes:

$$\frac{d^2e}{dt^2} \sim f$$

/79

In this way it is possible to obtain different types of analogies.

TABLE 7. MECHANICAL SYSTEMS AND THEIR ELECTRICAL ANALOGIES

	Mechanical Systems		Electrical Systems			
	Translation	Rotation	$i \sim f$ or x	$\frac{di}{dt} \sim f$ or v	$e \sim f$ or v	$\frac{de}{dt} \sim f$ or a
Constants	M C K	J C_t K_t	C_c G Γ	C_c G Γ	L R S	L R S
Variables	$\int f dt$ f $\frac{df}{dt}$	$\int \tau dt$ τ $\frac{d\tau}{dt}$	$\int i dt, q$ i $\frac{di}{dt}$	i $\frac{di}{dt}$ $\frac{d^2i}{dt^2}$	$\int e dt$ e $\frac{de}{dt}$	e $\frac{de}{dt}$ $\frac{d^2e}{dt^2}$
	x $\frac{dx}{dt}, v$ $\frac{d^2x}{dt^2}, a$	θ $\frac{d\theta}{dt}, \omega$ $\frac{d^2\theta}{dt^2}, \alpha$	$\int e dt$ e $\frac{de}{dt}$	e $\frac{de}{dt}$ $\frac{d^2e}{dt^2}$	$\int i dt, q$ i $\frac{di}{dt}$	i $\frac{di}{dt}$ $\frac{d^2i}{dt^2}$
Geometric Form of the Network Diagram	The diagram is based on the connections of the mechanical system.		The same form as that for the mechanical system.		Dual (if that is obtainable) of the form for the mechanical system.	

CHAPTER V

IMPEDANCE AND MOBILITY

5.1. Introduction

Use of the impedance and mobility method in the treatment of vibrational problems has increased greatly in recent years. By using this method it is possible to solve a complex system (with several degrees of freedom) by breaking it down into subsystems that can be analyzed more easily and then recombining these subsystems to find the response at a point or points of the real system. If the relative motion between two points of a complex system is known, it is possible to predict the effect of intercalating a component between these points; or if the force which acts on a branch is known, one can also predict the effect of intercalating a component in this branch. This method is useful for understanding the interactions which take place in a complex system and is based on the force-current analogies and takes advantage of the fact that there is a direct correspondence between a mechanical system and an electrical system. /80

In order to study this system with these methods, the force and motion characteristics must be known; they are dealt with in the form of complex expressions. These characteristics are defined as follows:

Force. A sinusoidal force can be represented as a rotating phasor (to differentiate it from a vector with magnitude and direction) in the complex plane, as is shown in Figure 22. The phasor has a length or magnitude F_0 and rotates in a counterclockwise direction with an angular velocity ω and at the same time t subtends an angle ωt with the real axis (+). The instantaneous force is the projection of the phasor on the real axis F cos ωt. The complex statement for force is: /81

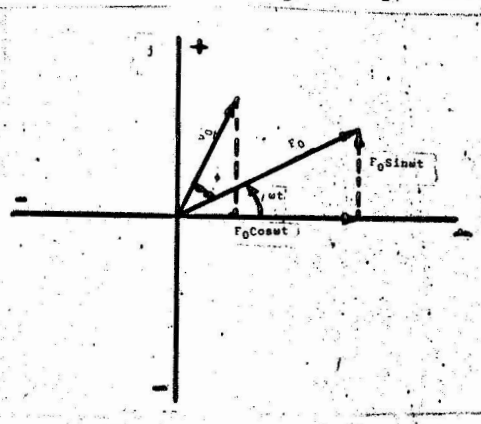


Fig. 22.

$$F = F_0 \cos\omega + j F_0 \sin\omega t$$

$$F = F_0 e^{j\omega t} \qquad (64)$$

Velocity. A sinusoidal velocity can also be represented as a phasor as is shown in Figure 22. When ν rotates at the same angular velocity ω as F_0, it can be represented as:

$$\nu = \nu_0 [\, \mathrm{Cos}(\omega t + \phi) + j\mathrm{Sin}(\omega t + \phi) \,] = \nu_0 e^{j(\omega t + \phi)} \tag{65}$$

where ϕ is the phase angle between F and ν. When F is taken as reference, ν can be described as: /82

$$\nu = \nu_0 e^{j\omega t} \tag{66}$$

Displacement. The displacement in the sinusoidal motion can be determined from the velocity, since $x = \int \nu dt = \int \nu_0 e^{j\omega t} dt$, one derives

$$x = \frac{\nu}{j\omega} \tag{67}$$

Acceleration. The displacement equal to acceleration can be found from the velocity, since $\ddot{x} = dv/dt$:

$$x = j\omega\nu \tag{68}$$

On the basis of these complex expressions of force and motion, impedance and mobility can be defined.

Mechanical Impedance. The mechanical impedance is defined as the expression of the excitation force that acts on a system at the resultant velocity of the system. If the velocity is measured at the point of application of the excitation force, the expression of force for velocity is the mechanical impedance at the point of application. If we designate this impedance by Z, the mathematical expression is:

$$Z = \frac{F}{\nu} \tag{69}$$

[See Reference (8)]

where F is the applied force and ν is the resultant velocity in the direction of force; both quantities are expressed as complex numbers, defined by Equations (64) and (65). When one takes F as a reference $\phi = \theta$, where θ is defined as the angle of impedance. /83

If one measures the velocity at another point in the system (in a direction different from the direction of the applied force) one can express the impedance as:

$$Z_{12} = \frac{F_1}{\nu_2} \tag{70}$$

This relation is known as "impedance of mechanical transference." F_1 is the sinusoidal force applied at a point in the system and v_2 is the sinusoidal velocity resulting at another point in the system; both are expressed as complex numbers.

Mobility. Mobility is defined as the inverse of impedance:

$$\mu = \frac{I}{Z} = \frac{\nu}{F} \tag{71}$$

where F and ν are defined by Equations (64) and (65).

In the pages which follow, we shall apply the impedance and mobility method to systems in translational motion, making clear the fact that these methods apply to translational systems as well as rotational systems but using the analogous parameters. The most frequently used quantities in these methods can be expressed in any consistent system of units; three such systems are given in Table 8.

The mobility and impedance methods permit a very complete analysis of the forces and motions which act on each part of a physical system. When it is necessary to analyze a system in only one or two points of interest, we introduce the concept of "black body" which determines these points of interest without a complete analysis of the total system. This concept is used for results of physical measurements in complicated systems and in which a total analysis is impossible. /84

TABLE 8. PARAMETER UNITS USED IN THE ANALYSIS OF VIBRATIONS IN IMPEDANCE AND MOBILITY METHODS

Quantity	Units		
	CGS	MKS	English Gravitational
Velocity ν	cm/sec	m/sec	in/sec
Force F	Dynes	Newtons	pounds
Mass M	Grams	Kilograms	$lb/sec^2/in.$
Rigidity K	Dynes/cm	Newtons/meter	lb/in.
Damping Coefficient	Dynes/cm/sec	Newtons/meter/sec	lb/in/sec
Impedance Z	Dynes/sec/cm Grams/sec	Newtons/sec/meter Kilograms/sec	lb/sec/in
Mobility μ	cm/dynes/sec sec/grams	Meters/newtons/sec Sec/kilograms	in/lb/sec
Potential P	Dynes/cm/sec Ergs/sec	Newton/meters/sec Joules/sec	in/lb/sec

5.2. Mechanical Elements of Impedance and Mobility

These elements are the same as those defined in Chapter I:

damping, springs, and masses. On the basis of the mathematical equation which defines these elements, one finds their impedance and mobility:

Damping. A damper or mechanical resistance is a device in which the relative velocity between its extreme points is proportional to the force applied at those points. For the case under consideration, or for an ideal element, the damping element is assumed to be completely rigid and massless. /85

In Figure 23 the velocity of Point A with respect to Point B is:

$$\nu = (\nu_1 - \nu_2) = \frac{F_a}{C} \tag{72}$$

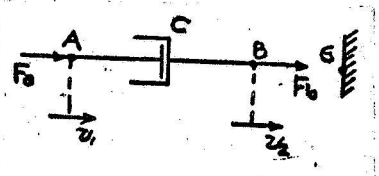


Fig. 23

where C is the proportionality constant, called damping constant. In order that there be a relative velocity ν as a result of the force on A, there should be a force of reaction on B. That is, it can be considered that the transmitted force F_b is equal to F_a. The velocities ν_1 and ν_2 are measured in relation to a fixed reference G; their difference is the relative velocity of the extreme points of the damper. Considering that the force applied on A is the sinusoidal force of Equation (64) and that B is the fixed point, the velocity ν will be:

$$\nu_1' = \nu = \frac{F_o\, e^{j\omega t}}{C} = \nu_o\, e^{j\omega t} \qquad (\nu_2 = 0) \tag{73}$$

It can be seen that the force and velocity phases rotate at the same angular frequency, that is, they are in phase. /86

The mechanical impedance for this element is:

$$Z_c = \frac{F}{\nu} = \frac{F_o\, e^{j\omega t}}{\frac{F_o}{C}\, e^{j\omega t}} = C \tag{74}$$

Or, the mechanical impedance of the damping element is the damping constant of that element.

The mobility will be the inverse of the damping constant:

$$\mu_c = \frac{I}{C} \tag{75}$$

Spring. A lineal spring is a device in which the relative displacement between its extreme points is proportional to the applied force. If one of its points is fixed, then:

$$x = \left(\frac{I}{K}\right) F \tag{76}$$

where 1/K is the proportionality constant known as deformation; its inverse K is the rigidity of the spring.

Substituting in Equation (76) the value of the sinusoidal force (considering that the applied force is sinusoidal) expressed in Equation (64),

$$x = \frac{F_o e^{j\omega t}}{K} = x_o e^{j\omega t} \tag{77}$$

To find the mechanical impedance of the spring, one derives the displacement in order to obtain the velocity, and substituting in Equation (69):

$$Z_k = \frac{F_o e^{j\omega t}}{j\omega F_o e^{j\omega t}/K} = \frac{K}{j\omega} = -\frac{j K}{\omega} \tag{78}$$ /87

The mechanical impedance of the spring depends on the rigidity of the spring, and the frequency is an imaginary number.

It can be easily demonstrated that the velocity is always in quadrature (90°) even though it is ahead of or lags behind the applied force.

The mechanical mobility of the spring will be the inverse of the impedance:

$$\mu_k = \frac{j\omega}{K} \tag{79}$$

Mass. In the element mass, the acceleration of the rigid body $\ddot{x}$ is proportional to the applied force:

$$\ddot{x} = \frac{I}{m} F$$

When a sinusoidal force is applied [Equation (64)] we have:

$$\ddot{x} = \frac{F_o e^{j\omega t}}{m}$$

To find the expression for mechanical impedance of the mass one integrates $\ddot{x}$ to find v, since F is known. Thus:

$$Z_m = \frac{F_o e^{j\omega t}}{F_o e^{j\omega t}/j\omega m} = j\omega m \qquad (80)$$

That is, the mechanical impedance of the mass is an imaginary quantity that depends on the frequency and magnitude of the mass. /88

As in the preceding page, it can be shown that the velocity of the mass has a phase lag of 90° with respect to the applied force.

The mechanical mobility of the mass is:

$$\mu_m = -\frac{j}{\omega m} \qquad (81)$$

If one knows the impedance and mechanical mobility of these three (passive) elements, one can analyze any system, as if one were dealing with an electrical circuit, by following the rules for circuits in series.

5.3. Mechanical Systems

The properties of mechanical systems which are used in the analysis of mechanical impedance and mobility are: points of force application, trajectories for force transmissions, and points of common velocity. The velocity at any point of the junction and the forces exercised on each element can be calculated by the correct application of Equations (69) or (71) if the impedance of each element of the system and the forces or velocities imposed on points of the system are known. In many applications, only the motion of a few points is of primary interest. It is in these cases that it is advantageous to combine groups of elements with simple impedances. Methods for calculating the impedance of such combinations of elements are described in the following section.

5.3.1. Parallel Elements.

To illustrate this combination, assume the two ideal elements connected in parallel of Figure 24. Both are compressed by the force F, for which reason both have the same relative velocity between their junctions. The force required to impose the velocity v on the spring is [see Equations (69) and (74)].

$$F_c = v_c$$

The force required to give the spring the same velocity is /89
[Equation (78)]

$$F_k = \nu \frac{K}{j\omega}$$

The total force F is, consequently:

$$F = F_c + F_k = \nu \left(C + \frac{K}{j\omega} \right)$$

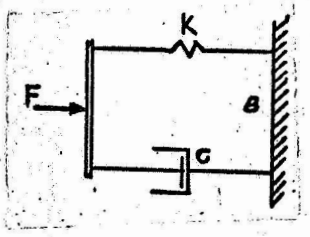


Fig. 24

And since $Z = F/\nu$, the impedance of the combined elements is:

$$Z = C + \frac{K}{j\omega}$$

Or, the total impedance of a combination of elements in parallel is the sum of the impedances of each element.

In general, for n groups of elements in parallel:

The applied force F will be equal to the sum of the forces in each element: /90

$$F = \sum_{i=1}^{n} \nu Z_i = \nu \sum_{i=1}^{n} Z_i \quad y \quad Z_p = \sum_{i=1}^{n} Z_i \tag{82}$$

where Z_p is the total impedance of the combination in parallel of the individual elements Z_i.

When the properties of the elements in parallel are expressed as mobilities, the total mobility of the combination in parallel is:

$$\frac{I}{\mu_p} = \sum_{i=1}^{n} \frac{I}{\mu_i} \tag{83}$$

5.3.2. Element Series.

In order to determine the total impedance of a number of elements joined in series, consider the grouping shown in Figure 25. The elements Z_1 and Z_2 are considered to be massless, since a mass always has one extreme connected to a stationary inertial point. However, the impedance Z_3 can be a mass. The relative velocities between the extreme connections of each element are shown in the figure by ν_a, ν_b, and ν_c; the velocities of the connections with respect to the fixed reference point G are indicated by ν_1, ν_2, and ν_3. Thus:

$$\nu_3 = \nu_c \; ; \; \nu_2 = \nu_3 + (\nu_2 - \nu_3) = \nu_c + \nu_b$$
$$\nu_1 = \nu_2 + (\nu_1 - \nu_2) = \nu_a + \nu_b + \nu_c$$

The force F is transmitted to all the elements; the impedance at point I is $Z_1 = F/v_1$. /91

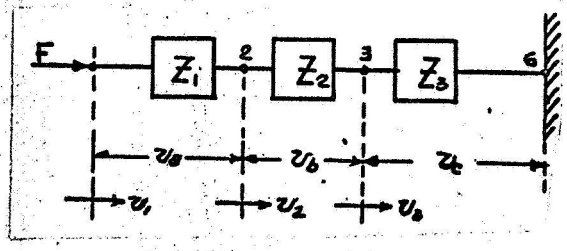


Fig. 25.

$$v_a = \frac{F}{Z_1} \quad v_b = \frac{F}{Z_2} \quad v_c = \frac{F}{Z_3}$$

The total impedance is defined by:

$$\frac{I}{Z} = \frac{F/Z_1 + F/Z_2 + F/Z_3}{F} = \frac{I}{Z_1} + \frac{I}{Z_2} + \frac{I}{Z_3}$$

Generalizing this principle for any number of massless elements joined in series, the total impedance will be:

$$\frac{I}{Z_s} = \sum_{i=1}^{n} \frac{I}{Z_i} \tag{84}$$

where Z_s is the total mechanical impedance of the elements Z_i connected in series.

The mechanical mobility of the elements connected in series will be:

$$\mu_s = \sum_{i=1}^{n} \mu_i \tag{85}$$

5.4. Equivalent Systems

When a connection point in a system of multiple elements is of interest, the methods described in the preceding section for the calculation of the impedance of combined elements can be useful in providing the impedance at this point. When a force is applied at the connecting point, the resultant velocity can be calculated from Equation (69). In general, this expression consists of real and imaginary terms. When one obtains the total impedance in this manner, the two terms can be considered as one resistant element (damper) connected in parallel with a purely imaginary or reactive element. An expression of mobility can be considered to represent these two elements; however, they are now joined in series. Each of these two representations has the characteristics of the multiple original system; a system thus represented is called an equivalent system. /92

The calculation of the total mechanical impedance or mobility requires calculation with complex numbers.

As an illustration of the preceding, let us find the total impedance at point A of the system shown in Figure 26, with the application of sinusoidal force F_A at the connecting point, and in this way we are in a position to find the velocity v which results from the application of this force.

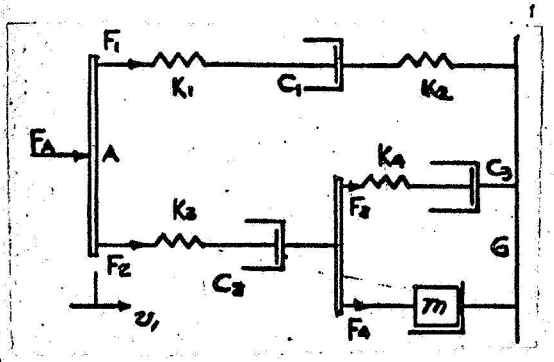


Fig. 26

The procedure to be followed is basically the following:

(1) When the elements in series (K_4, C_3) are in parallel with other elements (m), sum the mobility of the elements in series and convert the resulting expression to an impedance in order to be able to add directly the impedance of the elements in parallel.

(2) When elements connected in parallel (K_4, C_3, m) are connected in series with other elements (K_3, C_2), transform the impedance of the elements in parallel to mobility and add the mobility of the elements in series. /93

In agreement with (1) we sum the mobilities of K_4 and C_3 in order to obtain the mobility of the branch in B.

$$\mu_{B1} = \frac{I}{C_3} + \frac{j\omega}{K_4}$$

The impedance is:

$$Z_{B1} = \frac{I}{\mu_{B1}} = \frac{1/C_3 \quad j\omega/K_4}{(1/C_3)^2 + (\omega/K_4)^2}$$

$$Z_{B1} = \frac{I/C_3}{(I/C_3)^2 + (\omega/K_4)^2} - \frac{j\omega/K_4}{(I/C_3)^2 + (\omega/K_4)^2}$$

The system in B is then the combination in parallel shown in Figure 27A where $R(Z_{B1})$ represents the real part of Z_{B1} and (Z_{B1}) represents the imaginary part. The parallel impedance in B including the impedance of the mass m is

$$Z_B = \frac{I/C_3}{(I/C_3)^2 + (\omega/K_4)^2} + j\left[\omega m - \frac{\omega/K_4}{(I/C_3)^2 + (\omega/K_4)^2}\right]$$

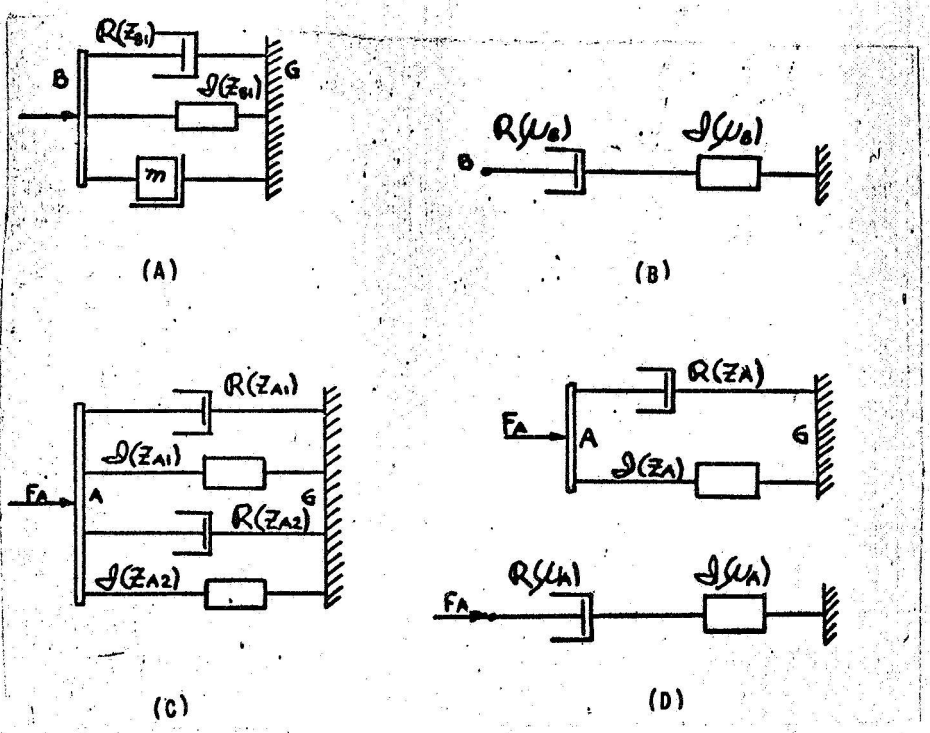


Fig. 27.

The corresponding mobility is:

$$\mu_B = \frac{\mathcal{R}(Z_B)}{|Z_B|^2} - j\,\frac{\mathcal{I}(Z_B)}{|Z_B|^2}$$

where $|Z_B|$ is the magnitude of Z_B represented in Figure 27B. Summing the mobilities of C_2 and K_3, the mobility present in A for this frame is: /94

$$\mu_{A2} = \mathcal{R}(\mu_B) + \frac{I}{C_2} + j\cdot\left[\mathcal{I}(\mu_B) + \frac{\omega}{K_3}\right]$$

In order to sum the effect of this branch, on the parallel branch (K_1, K_2, C_1), the two branches can be transformed into impedances in the following manner:

$$Z_{A2} = \frac{I}{\mu_{A2}} = \frac{\mathcal{R}(\mu_{A2})}{|\mu_{A2}|^2} - j\,\frac{\mathcal{I}(\mu_{A2})}{|\mu_{A2}|^2}$$

$$\mu_{A1} = \frac{I}{C_1} + \frac{K_1 + K_2}{j\omega}$$

$$Z_{A1} = \frac{I}{\mu_{A1}} = \frac{I/C_1}{(I/C_1)^2 + [(K_1+K_2)\omega]^2} + j\,\frac{(K_1+K_2)/\omega}{(I/C_1)^2 + [(K_1+K_2)/\omega]^2}$$

The impedances Z_{A1} and Z_{A2} can be represented by the four elements in parallel shown in Figure 27C. The total impedance in A is: /95

$$Z_A = Z_{A1} + Z_{A2} = \mathcal{R}(Z_{A1}) + \mathcal{R}(Z_{A2}) + j\,[\mathcal{J}(Z_{A1}) + \mathcal{J}(A_{A2})]$$

which can be represented by two elements in parallel (see Figure 27B). The expression of total mobility at A ($\mu_A = 1/Z_A$) can also be represented by two elements in series, as is shown in Figure 27B.

The expression for the force applied to the resultant velocity (total impedance) for a system of eight elements shown in Figure 26 will finally be expressed analytically by an impedance which has real and imaginary parts. The two elements are not necessarily damping elements, ideal springs, or ideal masses. The advantage of the representation of two equivalent elements is that the response of the complex system to an excitation frequency can be calculated easily, and the system can be represented by an ideal mass or an ideal spring and one resistance. In many cases, there are frequency ranges in which the complex system can be represented by one or two ideal elements. For example, in the preceding case, when $1/C_3 >> \omega/K_4$ (occurs when $\omega << K_4/C_3$), $Z_{B1} = C_3$), and the two elements can be represented by one simple resistance C_3. At higher frequencies $\omega >> K_4/C_3 \cdot Z_{B1} = K_4/j\omega$ and the impedance can be represented by a simple spring.

Another advantage of the representation of the equivalent system is that the effect of other elements intercalated at a point of interest can be considered. For example, when one provides a structure with a mass which has spring at the frequency of excitation, the possibility of resonance in the system exists. This may or may not be desirable, but in any case the size of the mass is very important. The quantitative determination of the response can /96 be obtained by combining the connected elements with the equivalent system.

In general, the characteristics of real masses, springs, and elements of mechanical resistance (dampers) differ from ideal elements in three respects:

(1) A spring can have a characteristic nonlineal force deflection, a mass can undergo plastic deformation with the motion and the force exercised by a resistance can be disproportional to the velocity.

(2) All of the materials have mass; therefore, springs and resistances have mass. Some deformation or spring effects are present in all of the elements. The energy can be dissipated in a system in various ways: friction, acoustical radiation, hysteresis, etc. Such an energy loss can be represented as a resistant component of the impedance of the element.

(3) The real elements can differ from ideal elements at high fre-

quencies or when there are many connected elements in the system, because the length of the element is comparable to the wavelength of the stress wave of the material.

It can be seen that in order to find the equivalent system for a complex system by the mobility and impedance methods, one applies exactly the same procedures as for electrical circuits, since the application of the first and second Kirchhoff laws to the theorems with the same conditions as for electrical circuits [Sic].

Superposition Theorem. If a mechanical system of lineal and bilateral[1] elements includes more than one vibration source, the force or velocity response at a point of the system can be determined by summing the response of each source by separately replacing /97 the other sources by their internal impedance.

The internal impedance of the source is the impedance present at the junction point, when the source does not provide energy. One important application of this theorem is seen when the vibration source is nonsinusoidal, since the source - whether it be of force or velocity - can be broken down into components of a Fourier series.

Reciprocity Theorem. If a force source operating at a particular frequency at some point (1) of a system of lineal and bilateral elements produces a velocity at another point (2), the source can be taken from (1) and substituted in (2), then the velocity in (2) will exist in (1), maintaining the impedances of all the points without change.

Thevenin Equivalent System. If a mechanical system having lineal and bilateral elements, containing vibration sources, and producing the total power for a load at some point at any particular frequency, the complete system can be represented for this frequency by a single constant force source F_c along with a single impedance Z_i connected to the load.

Norton Equivalent System. A mechanical system having bilateral and lineal elements with vibration sources and a connection with load can be represented at any particular frequency by a single source of constant velocity v_f in series with an internal impedance.

T and Π *Equivalent System.* Any mechanical system which has input and output connections and which is composed of lineal and /98 bilateral elements can be represented by a system of three elements having the configuration T or Π.

---

[1] A bilateral element is one in which the forces are transmitted in either of the two directions across its junction points.

5.5. Black Body Concept

It was mentioned in Section 5.1 that when it is necessary to analyze a system only at one or two points of interest, the concept of black body is introduced. This concept represents the system in two connections of passive elements joined to an inertial reference, having connected input and output as is shown in Figure 28.

When the elements in the system are lineal and bilateral and the sources of force are intercalated with the connections, the relations between the forces and velocities can be expressed as:

$$\left.\begin{aligned} F_1 &= Z_{11}\, v_1 + Z_{12}\, v_2 \\ F_2 &= Z_{21}\, v_1 + Z_{22}\, v_2 \end{aligned}\right\} \qquad (86)$$

where

$$Z_{11} = F_1/v_1|v_2 = 0,\quad Z_{12} = F_1/v_2|v_1 = 0$$

$$Z_{21} = F_2/v_1|v_2 = 0,\quad Z_{22} = F_2/v_2|v_1 = 0$$

known as impedance parameters. /99

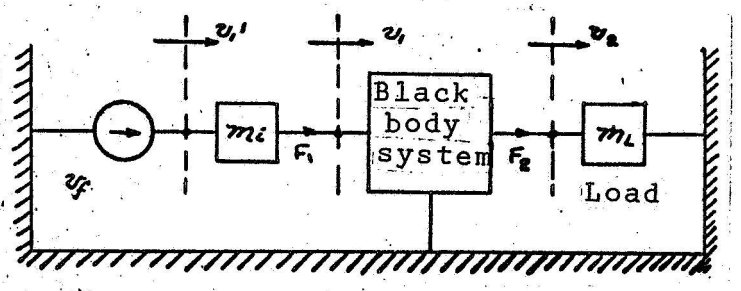


Fig. 28

When one knows the force and impedance parameters, one can find the velocities by resolving Equations (86).

The forces and the velocities of the system considered as black body can also be expressed as a function of the parameters of mobility; thus;

$$\left.\begin{aligned} v_1 &= \mu_{11}\, F_1 + \mu_{12}\, F_2 \\ v_2 &= \nu_{21}\, F_1 + \nu_{22}\, F_2 \end{aligned}\right\} \qquad (87)$$

where the parameters of mobility are:

$$\mu_{11} = \nu_1 / F_1 \,|F_2 = 0 \quad , \quad \mu_{12} = \nu_1 / F_2 \,|F_1 = 0$$

$$\mu_{21} = \nu_2 / F_2 \,|F_2 = 0 \quad , \quad \mu_{22} = \nu_2 / F_2 \,|F_1 = 0$$

Equations (87) are useful when the velocities in connections (1) and (2) are known.

CHAPTER VI

METHOD FOR CONTROL OF VIBRATIONS

6.1. Introduction

Up to this point we have seen the procedures which are followed /100
in an analysis of vibrations; it can be seen that there are sufficient methods for finding the magnitudes that enter into a vibration problem. In this chapter, we will describe a way of controlling these vibrations but, as was indicated in Chapter I, it is very difficult in designing to fabricate the moving components of the machine with 100% balance, due to the lack of homogeneity in the materials, errors in the geometric dimensions, and many other factors that enter into construction, even though the design be as exact as possible. This means that a body rotating about a certain axis exerts variable perturbations because the axis in which the body rotates does not coincide with the axes of inertia. Owing to the fact that the distribution of mass is not equilibrated or balanced, these perturbing forces can be of a high magnitude, especially when one deals with high velocity engines, and even more so if this velocity is close to or must of necessity surpass the critical velocity.

There exist different methods for controlling vibrations, /101
since balancing the moving components of the machine for their insulation from the surface where they are located (if that is stationary) or by means of dampers, in order to damp the vibrations (if this is not stationary), and, finally, detuning in order to avoid resonance. In the following section a method will be described which one follows to control vibrations by means of proper balancing of the rotating components.

6.2. Control of Vibrations by Balancing.

The balancing of a rotating component (rotor) is carried out in the majority of cases in a balancing machine, mechanically, and these machines give directly the amount of imbalance and the point where this imbalance is found.

The concept of balance is to offset by adding or removing weights the centrifugal forces that are not balanced by removing them or placing them in suitable positions so that the component remains as balanced as possible.

The method which is commonly used for balancing will be described, but first it should be mentioned that there are two types of imbalance: static and dynamic.

Static imbalance is that type of imbalance produced by eccentric masses; that is to say, the center of gravity of the rotating body does not coincide with the center of axis or rotation.

Dynamic imbalance, however, when the center of gravity coincides with the rotational center of the centrifugal forces for the different body masses, is not compensated in the same plane and therefore a resultant couple is produced.

Static balance can be effected simply by placing the rotor above two parallels; the heaviest part will tend to remain below, and the counterweight necessary to balance it should be diametrically opposed.

Dynamic balance alone can be brought about by centrifugal forces also produced by counterweights. /102

In the majority of imbalanced rotors, one finds these two types of imbalance; and dynamically, that is to say with tests in motion, one can learn the magnitude and the position of the counterweight (or the weight which must be removed) necessary to correct the two imbalances either separately or in combination.

6.2.1. Static Balance.

Assume a thin disc which rotates in a direction above two journal bearings. The center of the disc is displaced from the rotational axis by a distance e (Fig. 29).

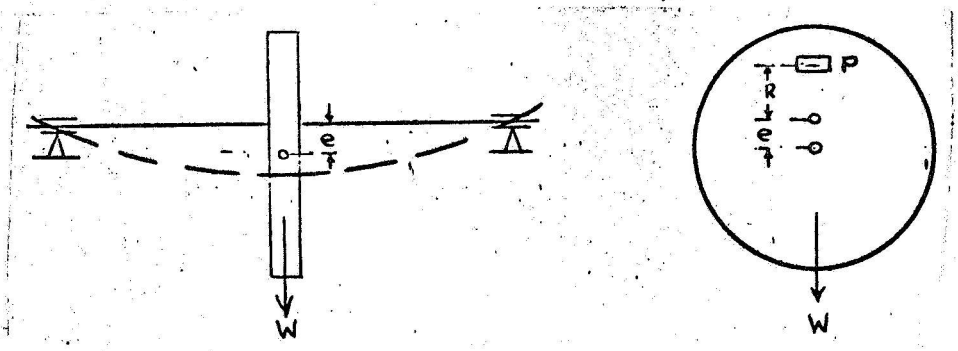


Fig. 29.

The imbalance can be considered as the product of the weight W of the disc multiplied by the eccentricity e. If the disc is made to rotate, the eccentricity will increase, bending the arrow

to the same extent that the centrifugal force increases, and will produce a vibration equal in frequency to the rotational. The arrow will take the form of a broken line. The imbalance is corrected by adding a weight P diametrically opposite at a distance r, so that:

$$Pxr = Wxe$$

If the disc is extended (Fig. 30) the counterweights P_1 and P_2 /103 located in the extreme plane and opposite to W, will balance the system when:

$$\left.\begin{aligned} P_1\ r_1 + P_2\ r_2 &= W\ x\ e \\ P_1\ r_1\ a &= P_2\ r_2\ b \end{aligned}\right|$$

Both cases are typical of static balance.

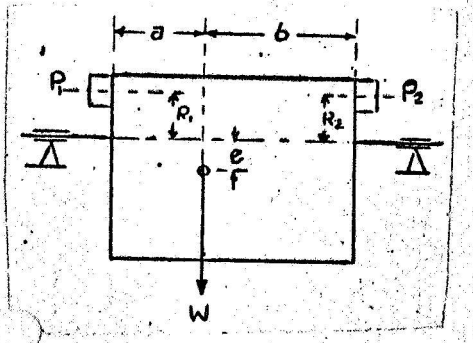


Fig. 30

6.2.2. Dynamic Balance.

If there are masses M which are not in diametric opposition, their centrifugal forces F will exert during the rotation a couple whose moment will be FL (see Fig. 31).

Let L be the separation of the two points at which force is applied. These forces appear when the rotor is in rotation and they increase with the velocity. They are characterized by the product F R L.

6.2.3. Combined Balancing. /104

We assume that P_e is the static imbalance and Pd_I and Pd_{II} are the imbalances in planes I and II (see Fig. 32.) produced by the centrifugal forces caused by dynamic imbalance. Pe can be broken down into Pe_I and Pe_{II} acting in planes I and II, respectively. The sum of the imbalances which act in each plane are P_I and P_{II} and the angle that they form between them will depend on the direction and magnitude of their components.

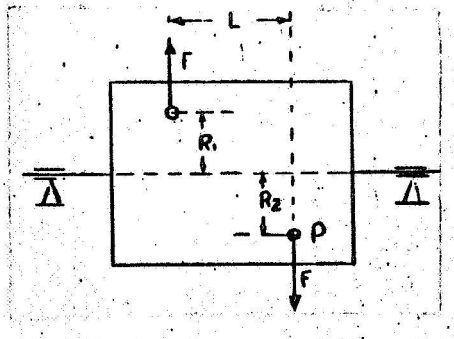


Fig. 31

In all of the balances it is

assumed that the magnitudes of the imbalances are proportional to the vibrations produced. In all of the calculations one works with forces measured in displacement units.

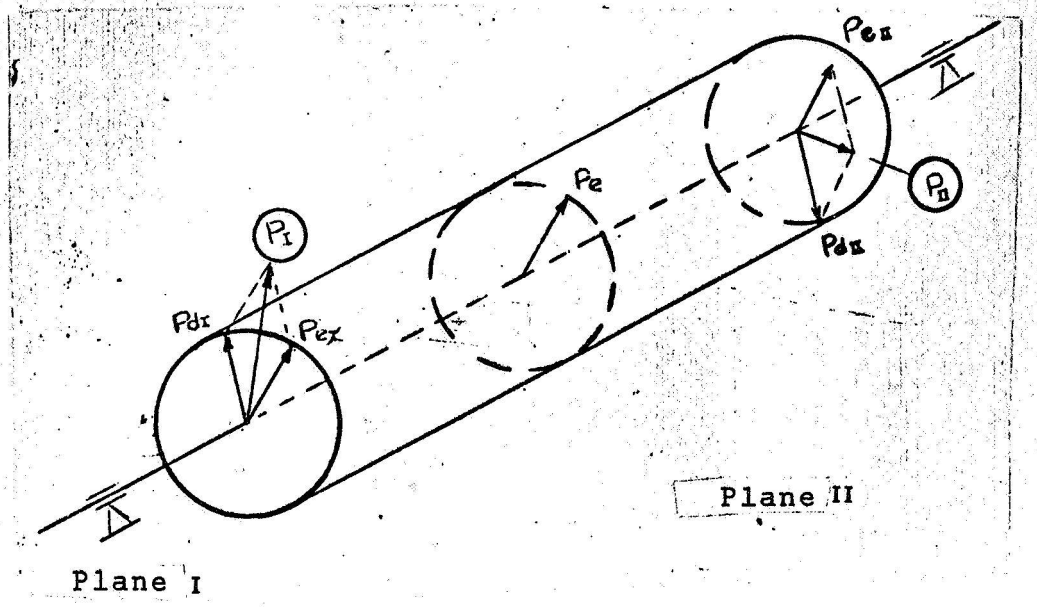


Fig. 32

Imbalance = Force = K × Displacement,
Let K be the proportionality constant. This is not absolutely certain but it yields practically acceptable results. /105

The direction of displacement never coincides with the direction of the force which produces it; that is to say, if one marks with a pencil the high point of the arrow (Fig. 33) of an imbalance rotor with the force P (weighted point), one will obtain a lag angle α which is the angle between the direction of force and the direction of displacement. This angle is called the lag angle.

All of the instruments for balancing determine only the high point.

The angle of lag α can be determined by making a run with test counterweights and calculating the effects, as will be seen below.

The counterweight necessary to balance the rotor should be placed to begin with diametrically opposite the weighted point. /106

The curve in Figure 34 shows the relation which exists between the angle of lag α and the velocity or, what amounts to the same thing, the relation between the high point and the weighted point upon varying the velocity.

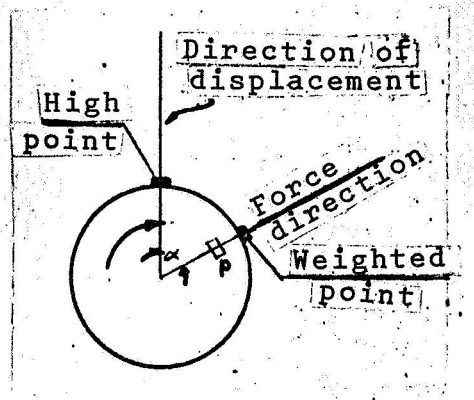
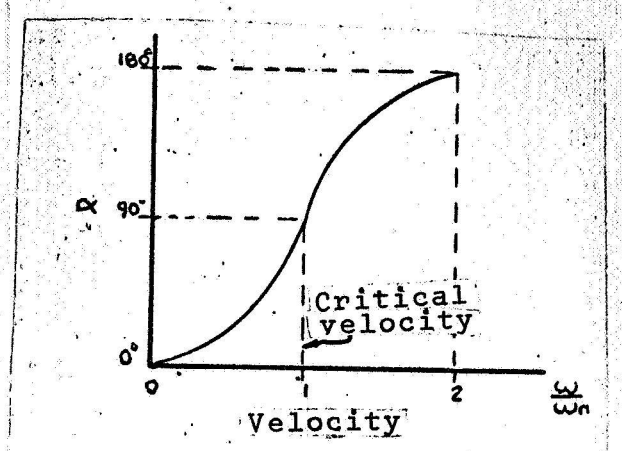


Fig. 33.



ω_n = Critical velocity ω = Velocity

Fig. 34.

The critical velocities can be determined by measuring the vibration of the machine in its entire range of velocity and making a vibration velocity curve as that which is shown in Figure 35. Analyzing this curve, one will obtain:

At 1200 RPM, where the natural frequency of the rotor enters into resonance with the rotational frequency, the direction of the vibrations on each side of the rotor will be in the same sense and can be eliminated or decreased by locating the counterweights as shown in Figure 30. This balancing should effect a velocity lower than the resonance, with two objects:

(a) to make less the action of the couple which produces the first critical dynamic velocity and /107

(b) to withdraw from the zone of instability which characterizes the rotor rotating at a critical velocity.

In the first critical dynamic velocity (at 1800 RPM) the directions of the forces of vibration found in the extremes of the rotor will be in opposite directions, and the counterweights should be placed opposite each other also. The balancing velocity should also be less than the resonance at this critical velocity.

Only in very extensive rotors are secondary and tertiary critical velocities encountered, and in general, machines are designed to work before the static velocity, after it, or after the primary dynamic velocity.

6.2.4. Balancing Process.

(1) Take a set of readings for the supports and the shafts at operational velocity and if possible for the whole velocity range of the machine.

(2) Determine the critical velocity.

(3) Make an analysis of the variations of vibration with the velocities and the pitch of the critical ones in order to define the velocities at which one should first balance the machine.

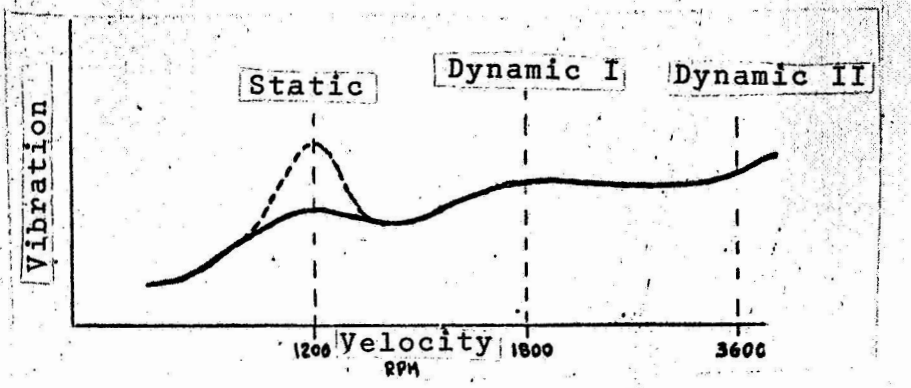


Fig. 35.

(4) Program the order of tests and balances to follow, that is to say, if it is necessary only to correct the dynamic or both at once. /108

(5) Place the test counterweights and calculate the angle of lag.

(6) Calculate the magnitude and the position of the counterweight or the final counterweights.

6.3. Control by Insulation

When the excited forces in a force vibration of a body cannot be eliminated, it is necessary to fall back upon some method of vibration insulation so that the reaction of the periodic force on the foundation or the armature that serves as a support is reduced. The usual method of insulation consists of employing some form of elastic suspension or insulator for the body. The function of an insulator is to reduce the magnitude of vibrational motion transmitted from the foundation to the system or to reduce the magnitude of the force transmitted from the system to its foundation.

The concept of insulation of vibrations is shown in Figure 36 illustrated by a system with one degree of freedom. This system consists of a rigid body which represents a device connected to a foundation by an insulator which has elasticity and ways of dissipating its energy. The system is unidirectional since it is restricted to motion in the vertical direction only. The functioning of an insulator can be known by the following response characteristics of the device-insulator system of Figure 36 for sinusoidal vibrations of constant state:

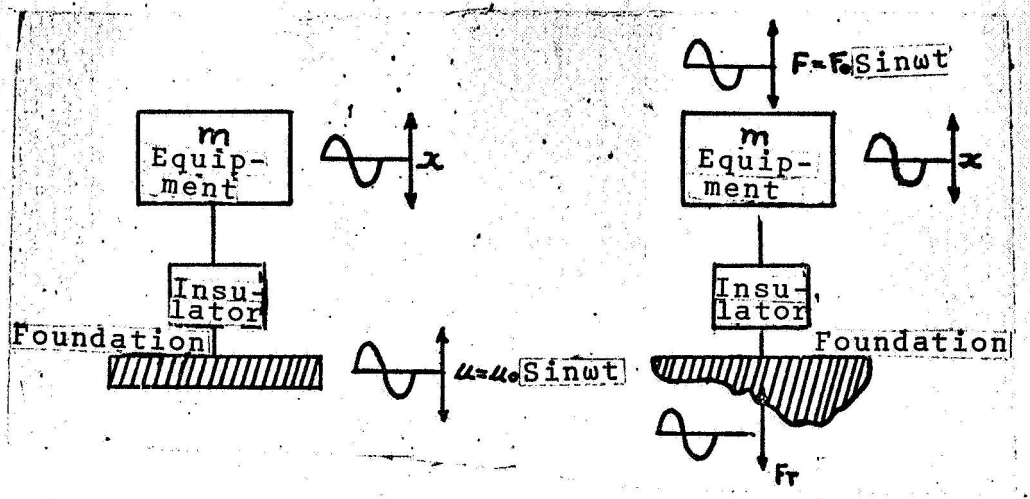


Fig. 36. Schematic Diagram for Insulation of Vibrations: (A) Insulations of Vibrations Where the Motion is Imposed on the Foundation and the Motion x is Transmitted to the Device. (B) Insulation of Vibrations Where the Force F Is Applied by the Device and the Force F_T Is Transmitted to the Foundation.

Absolute Transmissibility. Transmissibility is a means for /109
the reduction of the transmitted force or movement produced by an insulator. If the vibration source is an oscillatory motion of the foundation (excitation motion), the transmissibility is the expression of the vibration amplitude of the equipment at the vibration amplitude of the foundation. If the vibration source is an oscillatory force originating within the equipment (excitation force), the transmissibility is the expression of the force amplitude transmitted to the foundation at the amplitude of the excitation force.

Relative Transmissibility. Relative transmissibility is the /110
expression of the irregular relative amplitude of the insulator at the displacement amplitude imposed on the foundation. A vibration insulator reduces the vibration permitting a deflection of the insulator. The relative deflection is a way of opening (fair game) required in the insulator. This characteristic has significance only if an insulator to reduce the vibration transmitted from a foundation is used.

6.3.1. Types of Insulators.

The essential characteristics of an insulator are: elasticity in order to support the load, and ways of dissipating the energy. In certain types of insulators, the functions of supporting the load and dissipating the energy can be effected by a single element, example: natural or synthetic oilcloth. In other types of insulators, the elasticity for transporting the load can necessitate

sufficient characteristics for dissipating the energy, example: metal springs. The insulators with energy dissipating characteristics are known as dampers; and springs and dampers are considered separate elements.

There are different types of dampers, such as: Eddy current dampers, Coulomb friction dampers, and elastic dampers.

6.3.1.1. Elastic Dampers. The dampers of this type can be of different geometrical forms, depending upon the place in which they are located, and consist of two identically shaped plates placed parallel to each other and separated by a viscous fluid, as shown in Figure 37 (A).

In Figure 37 (A) is shown a damper which consists of two parallel plate planes with a viscous film between them; one or both plates has a velocity v_1, v_2 in its respective plane. The area of each plate is S, and the thickness of the viscous film is t. The coefficient of dynamic or absolute viscosity μ is the resistance offered by the viscous film, its thickness, and the difference in the velocities v_1-v_2. /111

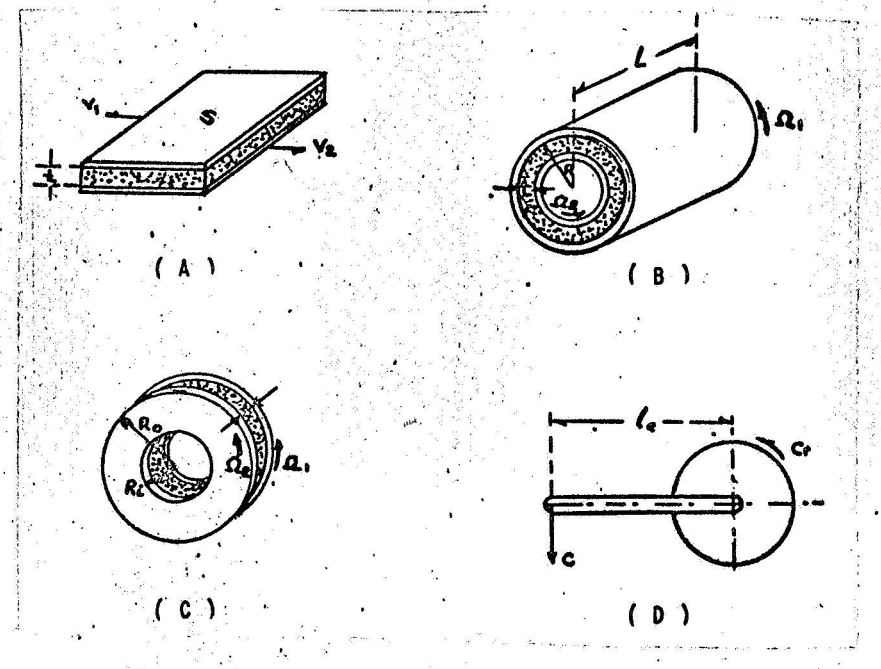


Fig. 37.

The dynamic viscosity is of fundamental importance in determining the fluid characteristics of dampers of the type illustrated in Figure 37 (A). Kinematic viscosity is tied in with the dynamic viscosity by mass density or: $\nu = \mu/\rho$, where μ = dynamic viscosity and ρ = mass density.

For purposes of damper design, it is most convenient to express the dynamic viscosity in lb/sec/in^2 units, the dynamic viscosity in /112
in^2/sec; the mass density therefore is expressed in lb/sec/in^2. In the metric system, the unit of dynamic viscosity, one dyne per square centimeter is known as a poise; the unit of kinematic viscosity, one square centimeter per second, is called a stoke.

The commercially available fluids are offered in units of centipoises (10^{-2} poises) or centistokes (10^{-2} stokes). The conversion factors from the metric to the English system are:

One centipoise = 1.45×10^{-7} lb/sec/in^2 (reyn)
One centistoke = 6.44×10^{-2} in^2/sec.

For the parallel plate damper shown in Figure 37 (A), the resistance or drag between the plates is directly proportional to the difference in the velocities (v_1-v_2). The damping coefficient is the force per unit of velocity:

$$c = \frac{S\mu}{t} \tag{88}$$

where S is the area of each plate, t is the thickness of the viscous film, μ is the coefficient of dynamic viscosity, and c is the damping coefficient. This value of c can be employed directly in the differential equation of motion for a system containing a damper.

In a practical sense, it is frequently more convenient to obtain the viscous damping of rotation for one cylinder within another. Then the film of viscous fluid is the same as that shown in Figure 37 (A) curved within the cylindrical form and placed between the two cylinders as shown in Figure 37 (B). The damping is a resistant couple and is directly proportional to the difference of radians per minute (Ω_1-Ω_2) of the concentric cylinders. The damping coefficient is:

$$c_r = \frac{2\pi R^3 l \mu}{t} \tag{89}$$

where these coefficients are indicated in Figure 37 (B). The damp- /113
ing coefficient c_r includes a torsional concept and is the resistance offered by the film per unit of differential velocity.

In another form of viscous damping in rotational motion, the viscous film is located between two parallel discs rotating on the same axis, as is shown in Figure 37 (C). The portion of the film adjacent to the perphery of the disc has a damping effect greater than that about the axis, owing to the fact that the differential velocity is greater, and for this reason, the moment of resultant force is also greater. The resultant couple of damping force can be determined by integrating the force upon the area of the disc.

The coefficient of viscous damping for this case is:

$$c_r = \frac{\Pi \mu}{2t}\left(R_o^4 - R_i^4 \right) \qquad (90)$$

where the constants are defined in Figure 37 (C).

For damping of lineal or translational vibrations a device can be constructed, by adding to any of the dampers shown in Figure 37 (B) or 37 (C) (depending on the case a) connecting rod, as is shown in Figure 37 (D). If the length of the connecting rod is l_c the coefficients of torsional damping given by Equation (89) or (90) can be transformed to the coefficient of translational damping c by applying the relation $c = c_r / l_c^2$.

6.3.1.2. Eddy Current Damper. A damping force directly proportional to the velocity can be attained by moving a short-circuited coil across a magnetic field. The direction of current flow in the conductor and the direction of motion of the conductor are both perpendicular to the direction of the magnetic field flow lines. A damper designed on this principle is shown in Figure 38. This employs a cylindrically shaped permanent magnet polarized as shown in Figure 38.

The upper and lower plates and a cylindrical core constitute the polar components which create a magnetic field which extends radially across an anular field iron. The conductor forms the circuit in this field iron and it is placed to move vertically. The damping force which acts on the conductor is opposed to its motion. /114

The coefficient of viscous damping is:

$$C = A \frac{B^2 l_s}{\rho}$$

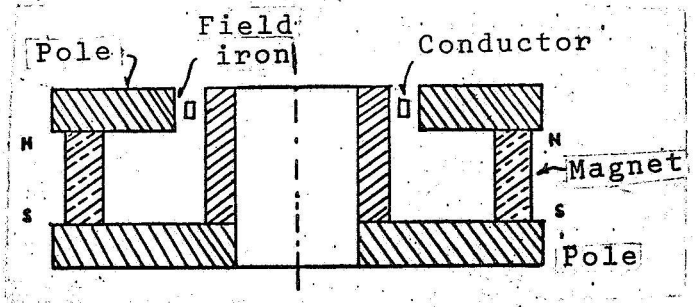


Fig. 38.

where B is the density of the fluid, l the length of the conductor, S is the area of the conductor cross section, and ρ is the resistivity of the conductor, and A is a constant which depends on the characteristics and units of the damper. The damping which can be obtained with this type of damper with a permanent magnet is useful only if the mass and natural frequency of the systems are relatively low.

6.3.1.3. Coulomb Friction Dampers. The magnitude of the Coulomb friction force is a constant independent of the velocity or dis-

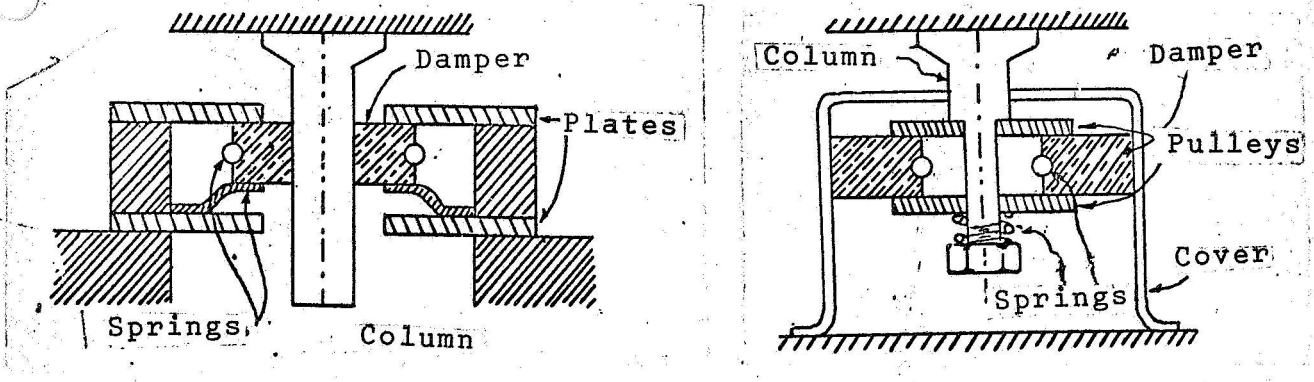
placement, but which has a direction which tends to oppose the relative motion across the damper. A Coulomb friction force generally /115
results from the relative motion of the solid components joined under pressure. The magnitude of damping force F_f is directly proportional to the friction coefficient η at the pressure P_n between the surfaces and the area of contact $F_f = \eta S P_n$.

The friction coefficient is a function of the material surfaces of contact where the relative motion occurs. Several values for friction coefficients for various materials in contact are given in Table 9. In general the static friction coefficient tends to be larger than the dynamic coefficient; for example, the resistance offered by the damper decreases somewhat after a relative velocity is attained.

TABLE 9. FRICTION COEFFICIENTS

Materials	Friction Coefficients
Bronze-steel	0.15
Steel-steel	0.15
Leather-steel	0.36
Nylon-metal	0.30
Teflon-metal	0.05

In constant state vibrations, the relative velocity equals zero twice per cycle. The effective friction coefficient decreases between the extremes.



(A) (B)

Fig. 39.

One of the problems which is encountered in the design of /116
friction dampers is the mechanical complexity involved in obtaining a structure capable of distributing a damping force in three directions. Dampers of this type are shown in Figures 39 (A) and (B). The first consists of a central column around which there is a

damping section supported by a contracting spring. The damper is localized between plates. A flat spring is placed between the damper and one of the plates to compress the damper against the other plate. The vertical motion damping is obtained by friction between /117 the column and the damper; horizontal motion is obtained by friction between the damper and the horizontal plates. The other damper, shown in Figure 39 (B), functions in a manner similar to the first. In the design for dampers shown in Figure 39, it is desirable that the springs have a relatively low rigidity and a relatively high deflection to obtain the normal pressure required.

(The normal pressure is the force exerted by the spring between the area of that portion of the spring supported by the plate.)

If the spring has these characteristics, the properties of the damper remain substantially unaffected by the use of the surfaces.

CHAPTER VII

DESIGN OF A MECHANICAL SYSTEM

7.1. Introduction

It has been indicated in the preceding chapters that one of /118
the principal problems which exists in machine components caused by the vibrations is breaking of components, rapid and excessive wear on the bearings, lack of lubrication, noises, etc.

This study is intended to anticipate vibrations in a vitally important component within internal combustion engines: the camshaft. The word "anticipate" has been used for reasons which will be explained in the following pages. In the design of a camshaft it is impossible to avoid resonance (critical velocity) - a condition most important for diminishing vibrations - due to the fact that machines of this type work at variable velocities within a determined range.

For this reason, the camshafts are designed for stresses such as those produced by fuel pressures within the cylinders and the vibrations produced by changes in velocity are removed by: sudden starts and stops and other factors, by means of devices which absorb these vibrations, or by placing dampers in one of the extremities of the crankshaft. In this way the damper absorbs a great quantity of vibrations, dissipating energy, and diminishing the amplitude of /119
these vibrations or - what amounts to the same thing - avoiding resonance. (Resonance develops when the excitation frequency is equal to the natural frequency).

Therefore, we shall proceed to calculate the excitation frequencies (critical velocities) of the camshaft of an internal combustion engine, in order to be in a position to design an adequate damper.

7.2. Calculation of Frequencies

For calculating the frequencies of the system that will be described below, one uses the Holzer Table method. In this procedure, one assumes a frequency beginning with obtaining a balance of couples and displacements step by step. The final exterior couple required to obtain balance is called the residual couple. If this is zero (or very close to zero), the assumed frequency is

one of the natural frequencies of the system. The mode associated with this natural frequency is identified by the number of sign changes for the amplitudes of the various rotors. A graph of residual couples against frequencies gives the natural frequencies of the system.

In many internal combustion engines the polar inertia of the rotors representing the connecting rods, crankshafts, and pistons are equal (or the difference is so small that they can be considered equal) as well as the torsional rigidity constants between these rotors. The advantage of this is that calculations are considerably reduced. If I is one of the polar moments of inertia (in dynes/cm/sec^2) and if $K_r = K$, one of the equal spring torsion constants (cm/dyne/rad), then dividing all the polar moments of inertia by I and all the torsional rigidity constants by K, we obtain a "unit system" dynamically similar to the original. Then, if ω_n is the natural angular frequency of the real system and ω_s that of the unit system, we have:

$$\omega_n^2 = \omega_s^2 \frac{K}{I} \qquad (91)$$ 7120

This unit form is very useful in the calculation of frequencies. Before beginning the sequence of calculations, it is necessary to know approximately the frequencies of the modes which are needed. Usually, no more than the two lowest are required. For an internal combustion engine, the following procedure is generally satisfactory:

Add all the polar inertias which are found in the engine (part of the system) and assume that 40% of this sum is placed at the extremity opposite the generator or flywheel (in crankshaft No. 1). Find the rigidities with reference to the flywheel (or the generator, if there is no flywheel connected) for crankshaft No. 1. If there is no flywheel, the system is reduced to a system of two rotors. If there is a flywheel, the system is a system of three rotors.

The system which we will consider consists of a diesel internal combustion engine having four cycles and eight cylinders in line; it is 22.86 cm (9") in diameter by 30.48 cm (12") in stroke. The inertias of each of the rotors, as well as the rigidity between them (already calculated) are indicated in the schematic diagram of the camshaft joint shown in Figure 40. From this figure we can show that the rotors 1, 4, 5, and 8 have greater inertia than the other four (referring to the eight cylinders) due to the counterweights which exist on these camshafts. The rotor represents the generator connected to this engine. This real system is transformed into a unit system by dividing the inertias between the larger corresponding to the cylinders or between 405×10^5 dynes/cm/sec^2 and the rigidity constants by 198×10^{11} dynes/cm/rad, values shown in Figure 40. This system can be transformed into a two-rotor system by applying the formula:

$$f_n = \frac{I}{2\Pi}\sqrt{\frac{I_1 + I_2}{I_1 I_2} K} \quad \text{cps} \qquad (92)$$

where

f_n = natural frequency
I_1 = Rotor 1 inertia = $0.40 \Sigma l_{lag}$
I_2 = Rotor 2 inertia = I_{10}
K = Total rigidity of the shaft (torsional rigidity)

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The sum of the inertial constants up to rotor 9 is:

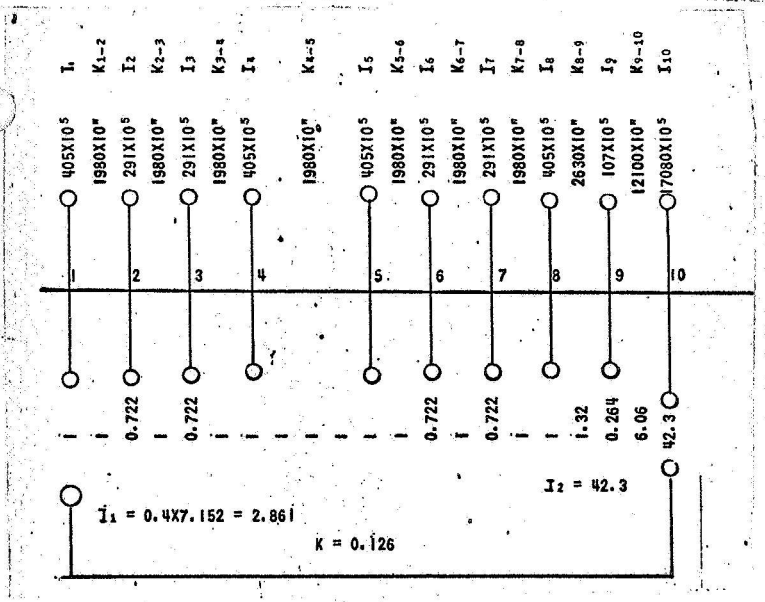
$$\sum_{i=1}^{n} I_i = 7.152$$

The rigidity of the shaft for rotor 10 is (total rigidity):

$$K = \frac{1}{\Sigma 1/K} = \frac{1}{7 + 1/1.32 + 1/6.06} = 0.126$$

On the basis of Equation (92) we can find a value of ω_s^2 as a first approximate:

$$\omega_s^2 = (2\Pi f n) = \frac{(I_1 + I_2)}{I_1 I_2} K = \frac{45.16 \times 0.126}{2.861 \times 42.3} = 0.047 \quad (\text{rad/sec})^2$$



Real system constants

Number of rotors

Unit system constants

2-Rotor system

Fig. 40

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Working from the value $\omega_s^2 = 0.047$, the calculations obtained by the Holzer method are grouped in the table shown in Table 10. The number of rotors and their inertial constants I, are placed in

alternate lines. The β values opposite the rotor number define the relative amplitude curve for the first mode, and the couples between the rotor numbers are the shaft couples between the corresponding rotors. The intermediate lines are designated as lines 1-2, 2-3, etc., the rigidity constants. For the first calculation $\omega_s^2 = 0.047$, the column $I\omega_s^2$ is calculated as the indicated product. For rotor 1 we will take an amplitude of 1.0. Then the inertial reaction of rotor 1 is $1 \times 0.047 = 0.047$; this value is placed in Column M corresponding to rotor 1. The couple in the portion of the shaft between rotors 1 and 2 (line 1-2) is equal to this inertial reaction and is located in Column M on line 1-2. This couple, divided by the rigidity 1.00 in section 1-2 of the shaft is the relative deflection (or deviation) of rotors 1-2 and it is located in Column M on line 1-2. This deflection remains 0.9530 for rotor 2. /123

This deflection multiplied by 0.0339 ($I\omega_s^2$ for rotor 2) gives the inertial reaction 0.0323 of rotor 2.

TABLE 10. SEQUENCE OF CALCULATIONS FOR THE FIRST MODE OF THE SYSTEM SHOWN IN FIGURE 40 ($\omega_s^2 = 0.047$)

Rotor Number	I	$I\omega_s^2$	Deflection β	Torque M, dyne/cm	
1	1.000	0.0470	1.0000	0.0470	
			0.0470	0.0470	1.00
2	0.722	0.0339	0.9530	0.0323	
			0.0793	0.0793	1.00
3	0.722	0.0339	0.8737	0.0296	
			0.1089	0.1089	1.00
4	1.000	0.0470	0.7648	0.0360	
			0.1449	0.1449	1.00
5	1.000	0.0470	0.6199	0.0291	
			0.1740	0.1740	1.00
6	0.722	0.0339	0.4459	0.0151	
			0.1891	0.1891	1.00
7	0.722	0.0339	0.2568	0.0076	
			0.1967	0.1967	1.00
8	1.000	0.0470	0.0601	0.0028	
			0.1511	0.1995	1.32
9	0.265	0.0125	- 0.0910	- 0.0011	
			0.0327	0.1984	6.06
10	42.5	1.9975	- 0.1237	- 0.2471	
			Residual Couple	Mr = - 0.0487	

The inertial reaction of rotor 2 summed for the couple in the /124
shaft section 1-2 gives the couple in section 2-3. And thus one finds successively all of the values, step by step. The final couple in the corresponding column corresponds to the residual couple. All of the additions and subtractions are carried out in the algebraic sense, for example:

On line 9 the deflection -0.0910 = 0.0601 - 0.1511.

The residual couple has the following physical significance:

If a simple harmonic couple of amplitude M_r and angular frequency ω_s is applied to rotor 10, then all the rotors will oscillate with the amplitudes given in column β and with the couples given in column M. If this sequence of calculations is done for sufficient value ranges of ω_s we will find all of the natural frequencies.

To find the first mode one assumes a new value for ω_s^2 obtaining for this value $M_r = 0.0253$. Interpolating for $M_r = 0$ between $\omega_s^2 = 0.047$ and 0.045 one obtains 0.0457. Table 11 shows the new values for $\omega_s^2 = 0.0457$, giving the conditions for the first mode with a frequency error lower than 1/1000.

TABLE 11. SEQUENCE OF CALCULATIONS FOR THE FIRST MODE OF THE SYSTEM SHOWN IN FIGURE 40 ($W_s^2 = 0.0457$) /125

Rotor Number	I	$I\omega_s^2$	Deflection β	Torque M, dyne/cm	K
1	1.000	0.0457	1.0000	0.0457	
			0.0457	0.0457	1.00
2	0.722	0.0330	0.9543	0.0315	
			0.0772	0.0772	1.00
3	0.772	0.0330	0.8771	0.0290	
			0.1062	0.1062	1.00
4	1.000	0.0457	0.7709	0.0351	
			0.1413	0.1413	1.00
5	1.000	0.0457	0.6296	0.0288	
			0.1701	0.1701	1.00
6	0.722	0.0330	0.4595	0.0152	
			0.1853	0.1853	1.00
7	0.722	0.0330	0.2742	0.0090	
			0.1943	0.1943	
8	1.000	0.0457	0.0799	0.0036	
			0.1499	0.1979	1.32
9	0.265	0.0121	- 0.0700	- 0.0009	
			0.0325	0.1971	6.06
10	42.500	1.9422	- 0.1025	- 0.1990	
			Residual Couple	Mr = 0.0019	

The frequency of the second mode corresponds to the condition in which there are two nodes in the machine shaft and has the value given by the equation:

$$\omega_s^2 = \frac{22}{n^2}, \quad n = \text{number of cylinders}$$

or $\omega_s^2 = 22/64 = 0.345$.

Through the calculation sequence one finds that this value is slightly higher so that M_r will be close to zero, or the calculated value is $\omega_s^2 = 0.360$; and for this value there is a new table (see Table 12). The natural frequencies of the real system can be found from Equation (91). /126

$$\omega_n^2 = \frac{1980 \times 10^n}{405 \times 10^5} \omega_s^2 = 4.9 \times 10^6 \omega_s^2$$

The natural frequencies in the first and second modes are:

First mode: $\omega_s^2 = 0.0457$; $\omega_n^2 = 224{,}000$; $f_n = 75.3$ cps

Second mode: $\omega_s^2 = 0.3600$; $\omega_n^2 = 1{,}765{,}000$; $f_n = 212$ cps

The critical velocities corresponding to these frequencies can be calculated from the equation $60\ f_n/q$ in RPM where q is the order number of the critical velocity[2] and f_n is the natural frequency of the modes in cps.

Values obtained in a diagram indicator.

For conditions of design it can be assumed that it is only necessary to calculate up to the second mode, and from Table 13 we can see that the most important critical velocities correspond to the first mode.

To find the dimensions of the damper, we will take into account the following specifications recommended by Lewis (see Reference No. 8).

(1) Place the damping element in the initial extremity of the crankshaft (before cylinder No. 1).

(2) Depending on the severity of the critical velocity for damping, assume a value between 5 and 25% of the total of the inertia in the machine for the floating component of the damper, and between 1 and 5 of the inertia outside the machine for the fixed component of the damper.

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[2] For footnote, see page 87.

TABLE 12. CALCULATION SEQUENCE FOR THE SECOND MODE OF THE SYSTEM SHOWN IN FIGURE 40

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Rotor Number	I	$I\omega_0^2$	Deflection β	Torque M, dyne/cm	K
1	1	0.360	1.0000	0.3600	
			0.3600	0.3600	1.00
2	0.722	0.260	0.6400	0.1660	
			0.5260	0.5260	1.00
3	0.722	0.260	0.1140	0.0296	
			0.5556	0.5556	1.00
4	1	0.360	- 0.4416	- 0.1580	
			0.3976	0.3976	1.00
5	1	0.360	- 0.8392	- 0.3020	
			0.0956	0.0956	1.00
6	0.722	0.260	- 0.9348	- 0.2430	
			- 0.1474	- 0.1474	1.00
7	0.722	0.260	- 0.7874	- 0.2945	
			- 0.3519	- 0.3519	1.00
8	1	0.360	- 0.4355	- 0.1570	
			- 0.3850	- 0.5089	1.32
9	0.265	0.095	- 0.00505	- 0.0048	
			- 0.0898	- 0.5137	6.06
10	92.3	15.23	+ 0.0343	0.5230	
			Residual Couple	M_r = + 0.0093	

TABLE 13. CRITICAL VELOCITIES FOR THE FIRST AND SECOND MODES

Order	4	4 1/2	5	5 1/2	6	6 1/2	7	7 1/2	8	12
1st Mode (rpm.)	1125	1000	900	820	750	692	643	600	562	375
2nd Mode (rpm.)	...	...	...	...	...	...	...	...	1670	1115
Harmonic Coefficients	29.5	23.5	18.5	15.0	12.0	8.8	6.5	5.5	3.9	1.4

2 The order number of a velocity is the number of complete oscillations of the system for one revolution of the camshaft.

(3) Make a table for calculation sequence in the unit form. /128

(4) Obtain a graph for various frequencies, to determine the fixed point or resonance point (intersection of the curves for the system without damping and with damping).

(5) Depending on the polar inertia, select a damper.

Taking into consideration the preceding recommendations, and carrying out the calculations in the unit form, one has:

Total inertia within the engine = 2.888 + 0.265 + 4 = 7.15
Inertia of the floating component of the damper = 10% × 7.1 =0.71
Inertia of the fixed component of the damper = 1% × 42.5 = 0.425

With these values and the values for the rotors of Figure 40, one constructs the new system shown in Figure 41.

Floating component of the damper /129
Fixed component of the damper

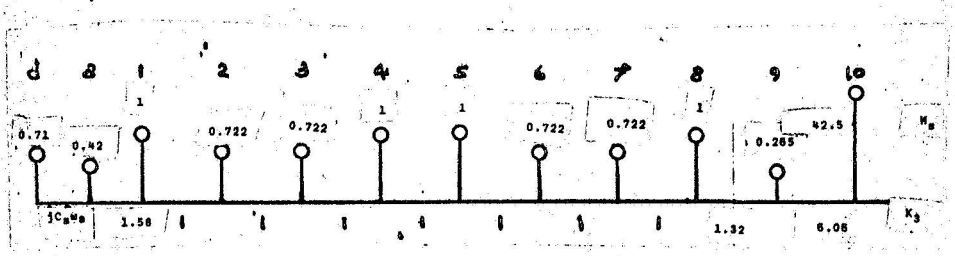

Fig. 41

With these values we can construct Table 14 for a value of $\omega_s^2 = 0.037$ as a first approximation.

$$\theta = \frac{4.39 + 0.703\ j/C_s}{-0.084+0.020\ j/C_s} \qquad |\theta|^2 = \frac{19.3 + 0.495\ /C_s^2}{0.0069+0.0004\ /C_s^2}$$

From Table 14 one finds for the case of free damping $\omega_s^2 = 0.037$. For the residual couple zero: /131

$$0.1474\ \theta + 5.121 = 0 \qquad \theta = -35$$

The larger couple in the shaft is found between rotors 9 and 10: $M_{9-10} = 1.57\ \theta = -1.57 \times 35$. The reciprocal is:

$$\frac{I}{M_{9-10}} = -\frac{I}{1.57 \times 35} = -0.01832$$

TABLE 14. CALCULATION SEQUENCE FOR THE SYSTEM SHOWN IN FIGURE 41 WITH A DAMPER AT THE EXTREMITY OF THE CAMSHAFT /130

Rotor Number	I	$I\omega_s^2$	β Inertia θ x	β Excitation	M, dyne/cm Inertia θ x	M, dyne/cm Excitation	K
10	42.5	1.570	1.0000		1.570		
			0.259		1.570		6.06
9	0.265	0.0113	0.741		0.0083		
			1.191		1.5783		1.32
8	1	0.037	-0.450		-0.0166		
			1.5617	1.000	1.5617	1.000	1.00
7	0.722	0.0267	-2.0117	- 1.000	-0.0538	-0.027	
			1.5079	1.973	1.5079	1.973	1.00
6	0.722	0.0267	-3.5196	- 2.973	-0.0935	-0.080	
			1.4144	2.893	1.4144	2.893	1.00
5	1	0.037	-4.9340	- 5.866	-0.1830	-0.217	
			1.2314	3.676	1.2314	3.676	1.00
4	1	0.037	-6.1654	- 9.542	-0.2280	-0.352	
			1.0034	4.324	1.0034	4.324	1.00
3	0.722	0.0267	-7.1688	-13.866	-0.1910	-0.371	
			0.8124	4.953	0.8124	4.953	1.00
2	0.722	0.0267	-7.9812	-18.819	-0.2130	-0.513	
			0.5994	5.450	0.5994	5.450	1.00
1	1	0.037	-8.5806	-24.269	-0.3170	-0.898	
			0.1785	3.510	0.2824	5.552	11.58
a	0.42	0.0155	-8.7591	-27.779	-0.1350	-0.431	
					0.1474	5.121	
					0.2824	5.552	
a+d	1.13	0.0418	-8.7591	-27.779	-0.3670	-1.161	
					-0.0846	4.391	
			-0.767 j/cs	-26.7 j/cs	0.1474	5.121	0.192 csj
d	0.71	0.0263	-8.7591	-27.779	-0.231	-0.730	
			+0.767 j/cs	+26.7 j/cs	+0.02 j/cs	+0.703 j/cs	
	ω_s = 0.192				-0.084	4.39	
					+0.02 j/cs	+0.703 j/cs	

Summing the damper d to a, the corresponding value is:

$$\frac{I}{M_{9-10}} = \frac{-0.0846}{4.391 \times 1.57} = -0.01225$$

Calculating the reciprocal of M_{9-10} for both frequencies (following the procedure in Table 14), the values indicated in Table 15 are obtained.

TABLE 15

ω_s^2	$\frac{I}{M_{9-10}}$	
	With Damping	Without Damping
0.035	-0.0261	0.000487
0.037	-0.01832	0.01225
0.038	-0.0137	0.0187

With these values a graph of the couple reciprocal as a function of frequency can be plotted (see Fig. 42); and thus one obtains the fixed point of resonance with a value of $\omega_s^2 = 0.0375$ for the frequency and a value of the couple reciprocal between 9 and 10 of 0.0160 in the unit system. In terms of the real system, this shows that with an optimal damper the vibrating couple will be 1/0.016 = 62.5 times for the excitation couple of a cylinder. /132

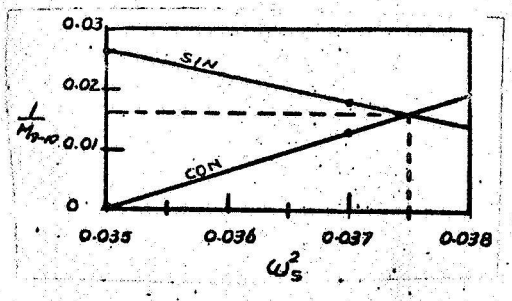

Fig. 42

The moment of excitation for any critical velocity is defined by the equation:

$$M_e = \frac{\pi}{A} D^2 \frac{L}{2} h \ (lb/in),$$

where D is the diameter of the cylinder in inches, L is the length of the stroke in inches, and h is the harmonic coefficient in a diagram indicator; for example, from Table 13 one can find the excitation couple corresponding to the critical octave:

$$M_e = 1490 \ lb/in = 383 \ h$$

In the engine, between rotor 9 × 10 one has:

$$\boxed{M_{9-10} = 93{,}200 \text{ lb/in}}$$

The new value for θ in the unit system is now /133

$$|\theta| = 62.5/1.570 = 39.8$$

The optimal damping constant C_s is found by considering the damper j $C_s\omega_s$ as a spring, where $\omega_s = \sqrt{0.037.} = 0.192$ and by continuing with the calculations in Table 14, the spring damping constant 0.192 $C_{si}j$ is placed in column K opposite to the damper D. Then the relative amplitudes in the "excitation" columns and "inertia" columns are the coefficients M/K:

$$\frac{0.1474}{0.192} = -\frac{0.767}{C_s} \quad j \qquad \frac{5.121}{0.192C_sj} = -\frac{26.7}{C_s}\, j$$

These values are placed in columns β and they contain the values β corresponding to rotor a, which gives the β values for the rotor d. The rotor amplitudes multiplied by 0.063 give the inertial couple reactions for rotor d. Summing this value for the pair between a and d, and making this sum equal to zero, we obtain the value for C_s,

$$|\theta| = \frac{4.39 + 0.703\ j/C_s}{-0.084+0.020\ j/C_s}$$

$$|\dot{\theta}|^2 = \frac{19.3 + 0.495\ /C_s^2}{0.0069+0.0004/C_s^2}$$

Solving for C_s and substituting θ = 39.8, one finds:

$$C_s = 0.131$$

The damping constant C in the real system is for the equation: $C^2 = C_s^2\ KI$

$$C = C_s\sqrt{405 \times 10^5 \times 1980 \times 10''} = 0.131 \times \sqrt{801900 \times 10^{16}}$$

$$C = 117.8 \times 10^8 \text{ dyne/cm/rad} = 10{,}400 \text{ lb/in/rad}$$

The polar inertia of the floating component (d) of the damper is:

$J = gI$, g = gravitational constant = 980 cm/sec^2

$J = 980 \times 405 \times 10^5 \times 0.71 = 271{,}500 \times 10^5$ dyne/cm^2

$= 9{,}800$ lb/in^2 /134

The dimensions for a damper of the Houdaille[3] type corresponding to the polar inertia obtained (9800 lb/in^2) are (see Fig. 43):

$R_2 = 8.25$ in. (20.96 cm.)

$R_1 = 4.125$ in. (10.48 cm.)

$b = 5.000$ in. (12.7 cm.)

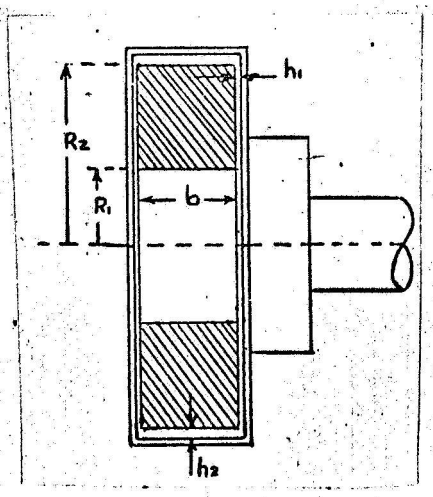

Fig. 43.

For this type of damper, the use of a liquid silicon with a viscosity between 20,000 and 60,000 centipoises is recommended.

The damping constant is given by the /135 expression:

$$C = 2\Pi\mu \left[\frac{R_2^3 b}{h_2} + \frac{I}{2} \frac{(R_2^4 - R_1^4)}{h_1} \right] \text{ lb/in/sec}$$

If one assumes a value between openings of $h_2 = h_1 = h$ and a viscosity of 40,000 centipoises, the value of the opening h derived from the preceding equation is:

$$h = \frac{2\Pi}{C} \mu \left[R_2^3 b + \frac{I}{2} (R_2^4 - R_1^4) \right]$$

where $\mu = 40{,}000$ centipoises $= 40{,}000 \times 1.45 \times 10^{-7}$ lb/sec/in^2

$\mu = 0.0058$ lb/sec/in^2 (reyn)

$$h = \frac{2\Pi}{10{,}400} (0.0058 \left[8.25^3 \times 5 + \frac{I}{2} (8.25^4 - 4.125^4) \right])$$

$h = 0.015$ in. (0.038 cm.)

---

[3] This type of damper is a flywheel mounted in a hermetic box with small openings around it. The box is filled with liquid silicon.

The damper for this system is that of Figure 43, it has the following characteristics:

R_2 = 8.25 in.	$h = h_1 = h_2$ = 0.015 in.
R_1 = 4.125 in.	J = 9.800 lb/in.
b = 5.000 in.	μ = 0.0058 reyn

CHAPTER VIII

SELECTION OF EQUIPMENT

8.1. Introduction

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In this chapter are described the principal types of instruments for measurement of vibrations and the selection of several of the most common types of instruments. Before describing the operating principle of these vibration measuring instruments, some terms which are employed in the classification of such instruments are defined:

A transducer (or sensor) is a device which converts vibrational motion into an optical image, a mechanical, or more commonly, an electrical signal which is proportional to a parameter of this motion.

A transducing element is the part of the transducer which transforms or converts the motion into a signal.

An accelerometer is a transducer whose output is proportional to the acceleration at the input.

A velocity transducer is a transducer whose output is proportional to the input velocity.

A displacement transducer is a transducer whose output is pro- /137
portional to the input displacement.

8.2. Classification of Instruments

In principle the vibrations are measured with reference to a fixed point in space: (For special purposes, the motion of a point can be measured with relation to another point. Then one terminal of the instrument is fixed to each one of the points.) This can be done by any of the different basic types of instruments:
(1) Fixed-reference instruments,
(2) Mass-spring instruments (seismic instruments).

In fixed reference instruments one terminal of the instrument is fitted to a point fixed in space and the other terminal is fitted to a point whose motion is to be measured (mechanical, optical, or electrical).

In the mass-spring instruments, the base is fitted to the point at which vibration is to be measured. The movement at the point is derived from the relative mass of the base.

8.2.1. Principle of Fixed Reference Instruments.

Figure 44 shows the principle of operation of the instruments for measuring the relative motion of a vibrating component and the fixed reference. By fitting a scale to the fixed reference and an index or indicator to the vibrating component, as is shown in Figure 44 (A), one can determine visually the limits of motion for the vibrating component. If the indicator moves beyond the scale, we can determine the displacement from peak to peak by measuring the length of the inscribed line. More information can be obtained if we substitute for the scale a rotating cylinder, as is shown in Figure 44 (B); as this cylinder rotates, the indicator for the vibrating component will mark a continuous line, which represents the time of relative displacement of the vibrating component for the fixed reference.

There are various methods for carrying out the functions /138
shown in Figure 44. Generally, the displacement connected with the vibrational motion is relatively small; therefore, it is necessary to amplify it in order to obtain sense and accuracy in the measurements. These methods vary depending upon the construction and the type of instrument.

Fixed reference instruments can be classified according to the type of measurement as follows:

(a) Visual displacement indicator of the wedge type.
(b) Microscopic measuring device - visual technique.
(c) Stroboscope - visual technique.
(d) Camera of the frame type.
(e) Line photography.
(f) Mechanical deformation recorders.

8.2.1.a. Visual Displacement Indicator of the Wedge Type. With /139
this type of instrument one obtains a quasi-optical amplification

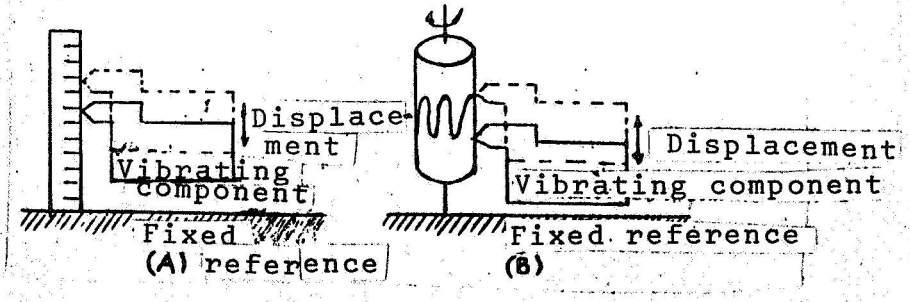

Fig. 44.

and the measurement of the displacement amplitude in almost any range by drawing a wedge of appropriate size on a paper. The drawing or a photographic reproduction is affixed to the system in which the vibrations are to be measured by cementing or by pressure-sensitive adhesive tape. The indicator with a persistent vision to produce the visual impression of a shaded triangle with gray angles, the position of the apex of the triangle gives the value for the vibrational amplitude. The effective amplification (magnification) which can be obtained is the expression of the height of the triangle to the length of its base.

8.2.1.b. Microscopic Measuring Device - Visual Technique. The normal unaided eye sees an object with greatest clarity at a distance of approximately 25 cm; any decrease of this distance tends to make the object larger in size and less precise in form, owing to the inability of the eye to adjust. The function of the microscope is to bring the image into acute focus while preserving the enlarged size. This is done by means of a system of lenses whose complexity varies with the number and type of elements (see Fig. 45) used in conjunction with the microscope: the ocular reticulum, the ocular micrometer, and the guage.

The advantages of this method of measuring vibrations which other measurement techniques do not have are:
(1) One can measure displacement amplitudes between 0.0001 inches and 1 inch. /140

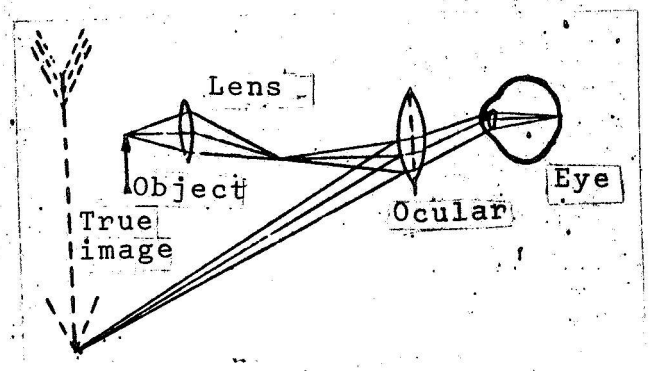

Fig. 45

(2) Only one source of light is needed as a source of energy.
(3) Frequency limits are not included in this technique.
Additional advantages include simplicity, accuracy in calibration, and the absence of load on the vibrating surface that is to be measured. The disadvantages of this method include the necessity of placing an extremely rigid support on the vibrating surface, the necessity for constant vibrational motion, and the fact that this method is inconvenient and long.

8.2.1.c. Stroboscope System - Visual Technique. The study of displacement amplitude is greatly facilitated by stroboscopic light, by means of which the body is seen under illumination of cyclical pulsed light. A common technique consists of adding a "cat hair" (for example, a "white hair") to the vibrating surface, as is shown in Figure 46. The microscope is placed at a right angle to the "cat hair" and with the point this creates the microscopic field. The stroboscope illuminates the "cat hair" against a dark field by adjusting the observed vibration frequency in such a way that the sweep of the "cat hair" across the microscope can be seen. Since the measured sweep of the "cat hair" is observed against a reticulum in the ocular of the

microscope, a firm support should be distributed independent of the vibrating system.

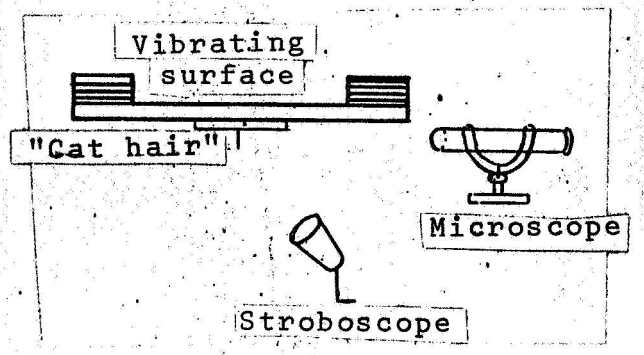

Fig. 46

In essence, all of the /141
stroboscopic methods permit a determination of observed motion frequency and the method is the same as that provided by the intermittent illumination of the stroboscope; in this determination it is not necessary to place a load on the vibrating system. An important advantage of this method of observations is that it is possible to see vibrations or rotations at low velocities.

8.2.1.d. Frame-Type Camera. This type of camera differs from the line type in that one takes a series of separate photographs of the phenomenon being photographed, and depends upon the persistence of vision for the illusion of continuity when the film is projected. The frame type cameras can use any of the following techniques; (1) the film can remain motionless, with successive changes in the registered event in superposition through flashes of intermittent illumination, or (2) the film can move either intermittently or continuously with successive proportionate images by shutters or rotating prisms. The camera with intermittent motion of the film are not used for expressions which exceed 200 photographs per second. The appropriate selection and the use of auxiliary equipment (for example, markers, indicators, stroboscopes, exposure meters, etc.) in many cases is of great importance for the production of satisfactory results.

8.2.1.e. Line Photography. With this method it is possible to obtain displacement time curves which can include amplitudes of various lengths with precision to hundredths of inches and thousandths of seconds. The image is engraved on the film by continuous undulating lines against a contrasting background. This can be described vs. time as a single Cartesian displacement point. This point is obtained from the reflection of light from a point of the white picture or of a small steel ball or a fine wire engraved on a dark /142
background of the surface under study. These points are small in order to ensure the reflection of the source and are located in the visual field compatible with the focal depth, so that the motion under study is at a right angle with the motion of the film. The displacement scale depends on the distance from the point to the camera and on the focal length of the lens.

8.2.1.f. Mechanical Deformation Recorders. The mechanical deformation recorders (also known as stress recorders) measure the deformation by registering the relative motion between two points on the surface of a structure. One can obtain a time-resolved history of

the deformations (or stresses) by recording the motion in an appropriate way. However, this type of device is not necessarily a fixed reference instrument; it is included here for convenience. The advantage of this type of instrument is that it provides a direct measurement of the stresses on a structure and of the forces acting on it. Since this does not depend on the response of a mass-spring system, it can be used to measure the vibrations below zero frequency. The upper frequency limit is determined by the inertia of the elements and links.

A deformation recorder consists of a resistance made of very fine wire (approximately 0.001 in. in diameter corresponding to a third of a human hair), which is firmly fastened to a paper, and at its extremities are soldered the connecting electrodes which carry the signals to the measuring apparatus.

The measuring device is cemented firmly to the surface of the test object in such a way that after the cement used to affix the measuring device has dried, it carries out the motions of the object exactly, undergoing the same deformations as the object. In case of an extension, the resistance of the measuring devices increases, while the resistance decreases with compression.

The principal advantages of this type of measuring devices /143
are:
(1) The dimensions are small, therefore measurements can be verified in places which are normally inaccessible;
(2) The weight is insignificant, therefore the operating conditions are not affected;
(3) The stress measuring devices are highly sensitive, reasonably stable, and permit a static calibration;
(4) When these are used in conjunction with an adequate indicator, these devices can be used for dynamic measurements at quite high frequencies.

8.2.2. Principle of Mass-Spring Instruments.

In many applications, for instance in vehicles in motion, projectiles, etc., it is impossible to establish a fixed reference to measure the vibrations; for this reason mass-spring instruments known as mass-spring transducers are employed in these cases. One transducer of this type is shown in Figure 47. It consists of a mass M suspended from a covered transducer by a spring of rigidity K. The motion of the mass within the cover is damped by a viscous-type damper or by an electrical current with a damping coefficient C. The covered transducer is located in a moving part (where the vibrations are to be measured); by means of a convenient scale we obtain the amplitude of the vibrations at the time that mass m registers its movement δ; the relative motion δ/u provides the real amplitude of the vibrations. Or the motion u is derived from the relative motion δ between the mass m and the cover a.

The mass-spring transducers can be adapted for measurements of displacement, velocity, or acceleration, depending on the natural frequency of the transducer and the transducer element.

The mass-spring transducers are in general those most used and are classified as follows:

(a) Vibrographs
(b) Vibrometers
(c) Vibrometric frequency indicators
(d) Seismograph
(e) Torsional vibration recorders.

The first two types (a and b) differ in their way of measuring /144
the vibrations, since it (a) records the vibrations on an appropriate paper by giving a time-resolved history of displacement while (b) gives a reading on a paper with a graduated scale; at one determined instant (does not continue). There are available on the market numerous types of these measuring instruments which differ in certain respects according to the manufacturer; here the best known are given;

(1) Askania universal vibrograph
(2) Askania manual vibrograph
(3) Cambridge universal vibrograph
(4) Davey manual vibrograph
(5) Korfund manual vibrograph
(6) Westinghouse LE vibrograph
(7) General Electric recording vibrometer
(8) Westinghouse vibrometer
(9) Davey vibrometer

Among the vibrometric frequency indicators one very well-known type (c) is the feather type. These feathers are supported only at one end in cantilever, as shown in Figure 48; generally, these feathers have a small mass at the free end, in such a way that the natural frequency of the feather is given by the expression:

$$fn = \frac{1}{2\pi} \sqrt{3EI/l^3 \left(m_2 + 0.23\, m_1 \right)} \quad \text{cps}$$

letting E be the modulus of elasticity, I the moment of inertia of /145
the section, l the length of the feather, m the mass of the feather, and m_2 the mass at the free end of the feather.

To determine the vibration frequency, these instruments are placed on the vibrating surface and by means of a previously graduated scale around the feathers will indicate the vibration frequency by means of the oscillations of the feathers.

This type of instrument has the advantage of being small, portable, and not requiring a source of energy for its operation.

The seismographs (d) are mass-spring instruments designed for seismographic measurements in the field. They consist of one or more

seismometric velocity transducers, of a galvanometric oscillographic photographic recorder, and an energy source for their operation.

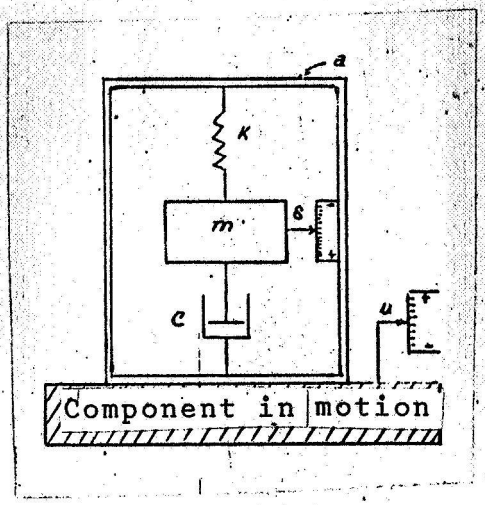

Fig. 47.

These seismometers are sensitive velocity transducers which use a magnet supported by a spring, like the mass of the mass-spring system, and the coil attached to the cover. These seismometers can be used for measurement either of horizontal or vertical vibrations.

The galvanometric seismometric system (without intermediate amplification) is sufficiently sensitive to detect earth movements on the order of 10^{-7} cm.

Due to the size and weight of these instruments, they are not used to measure vibrations in vehicles and machinery; they also require a source of energy. /146

Recorders of torsional vibration (e) as their name indicates are rotational displacement transducers which consist of a flywheel of energy pulsed across an elastic junction by the shaft being tested. An indicator needle is connected to the seismic mass and to the shaft across a system of mechanical links. Any angular displacement between the seismic mass and the rotating shaft is reflected by the indicator needle which records the motion on a special paper. In this way one obtains a polar diagram of the torsional vibrations by reading the amplitude on the transparent scale. The polar diagram permits the direct determination of the order of torsional vibration (the number of oscillation cycles per revolution of the shaft), but does not provide information concerning the frequency.

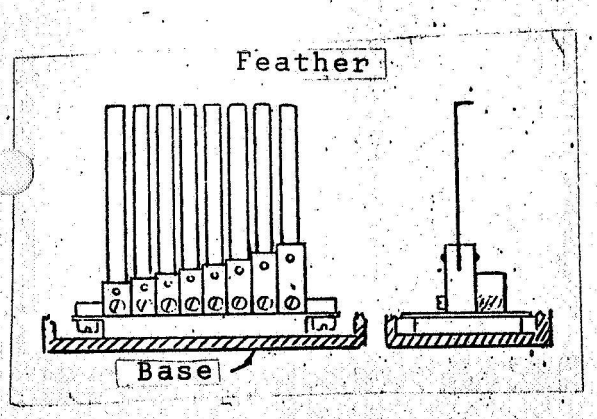

Fig. 48

CHAPTER IX

CONCLUSIONS

The systems with one and two degrees of freedom can be solved /147
easily by analytical methods.

For systems with a larger number of degrees of freedom, solution by these methods results many times in considerable and laborious work. In these cases one employs estimation or approximation methods, which can be solved very quickly saving time and without which the accuracy obtained is very distant from reality.

With these approximations one can elaborate programs for solution in a digital computor.

When it is necessary to analyze a system in which it is required to know its operation, this can be done by constructing a scale /148
model (taking into account the fact that time and expense are compensated by the results to be obtained).

In other cases, when it is desired only to analyze various systems and to know the results, one can make use of analogies. Since with this procedure, analyzed in an analog computer, the results can easily be found (by means of graphs) simply by changing and varying parameters.

The impedance and mobility method is appropriate for solution of vibration problems by analysis as though they were electrical circuits; all of the procedures which are followed are used in their solution.

In the manufacture of machine parts, such as electric motor rotors, flywheels, camshafts, etc., it is always necessary to make use of balanced machines, since in the process, it is not possible--nor does it reward the investment--to obtain 100% balanced and homogeneous parts.

One of the principle methods for controlling vibrations in machine components consists of decreasing vibration by placing damping devices in the machine. There are a great variety of these devices on the market.

Vibration problems exist in almost all the moving parts of a machine; in some cases this is desirable, but in many other cases it is to be avoided.

These problems constitute an extensive study, from design of packing for machine shipment to the elimination of noises and motions which produce human discomfort.

There are on the market numerous devices to detect and measure vibrations--from mechanical to electronic types.

The design for a vibrating system tends to become more complicated as the degress of freedom increase, and in all cases, they follow norms laid down beforehand.

In design of shafts with a limit of reasonable security, the criterion of critical velocity should take precedence in design over the criterion of stress, and the former should determine the diameter of the axle or shaft.

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